Part III

第三部分

Asymptotically Safe Quantum Gravity

渐近安全量子引力

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罗伯托·佩尔卡西

The Functional Renormalization Group in Quantum Gravity

量子引力中的泛函重整化群

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摘要

The gravitational asymptotic safety program envisions a high-energy completion of the gravitational interactions by an interacting renormalization group fixed point, the Reuter fixed point. The primary tools for investigating this scenario are functional renormalization group equations, foremost the Wetterich equation. This equation implements the idea of the Wilsonian renormalization group by integrating out quantum fluctuations shell-by-shell in momentum space and gives access to the theory's renormalization group flow beyond the realm of the perturbation theory. This chapter gives a pedagogical introduction to the gravitational asymptotic safety program with a specific focus on clarifying conceptual points which led to confusion in the past. We provide a step-by-step introduction to the Wetterich equation and its most commonly used non-perturbative approximations. This exposition also introduces recent developments including the minimal essential scheme and N -type cutoffs. The use of the Wetterich equation in explicit computations is illustrated within the Einstein-Hilbert truncation which constitutes the simplest non-perturbative approximation of the gravitational renormalization group flow. We conclude with a brief summary and comments on recent developments originating from other quantum gravity programs.

引力渐近安全方案设想通过一个相互作用的重整化群不动点——罗特不动点，完成引力相互作用的高能完备化。研究这一方案的核心工具是泛函重整化群方程，其中最具代表性的是韦特里希方程。该方程实现了威尔逊重整化群的思想: 在动量空间中逐层积分掉量子涨落，并且可以研究微扰论范围之外的理论重整化群流。本章为引力渐近安全方案提供教学式介绍，专门聚焦澄清过去引发困惑的概念问题。我们逐步介绍韦特里希方程及其最常用的非微扰近似。本次阐述也介绍了包含最小本质方案和 N 型截断在内的最新进展。我们在爱因斯坦-希尔伯特截断中展示了韦特里希方程在具体计算中的应用，该截断是引力重整化群流最简单的非微扰近似。最后我们对源自其他量子引力方案的最新进展做了简要总结与评论。

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Keywords

关键词

Quantum gravity - Asymptotic safety - Renormalization group - Wetterich equation - Reuter fixed point - Einstein-Hilbert truncation - Phase diagram

量子引力 - 渐近安全 - 重整化群 - 韦特里希方程 - 罗伊特不动点 - 爱因斯坦-希尔伯特截断 - 相图

Introduction

引言

Our theoretical understanding of nature rests on two pillars. The electroweak and strong force and their interactions with the elementary particles are described by the standard model of particle physics. This theory is formulated as a relativistic quantum field theory in Minkowski space. The description of gravity is provided by general relativity, a classical field theory which encodes the gravitational interactions in the dynamics of spacetime. Conceptually, these theories are on very different footing and the construction of a framework unifying gravity with the laws of quantum mechanics is one of the key open questions in theoretical high-energy physics to date.

我们对自然的理论认知建立在两大支柱之上。电弱力、强相互作用及其与基本粒子的相互作用由粒子物理标准模型描述，该理论是 formulated 为闵氏空间下的相对论量子场论。引力则由广义相对论描述，这是一种经典场论，将引力相互作用编码在时空动力学之中。从概念上看，这两种理论的基础截然不同，构建一个将引力与量子力学定律统一的框架，至今仍是理论高能物理学中核心的开放性问题之一。

An important insight along these lines is that the quantization techniques successful in the case of the standard model of particle physics do not extend to gravity in a straightforward way: the perturbative quantization of general relativity leads to a perturbatively non-renormalizable quantum field theory with new infinities appearing at every order in the perturbation theory [1-4]. This has led to the advance of several physics principles which deviate from the principles of the continuum quantum field theory in more or less radical ways; see [5, 6] for recent nontechnical accounts.

沿这一方向的一个重要认知是: 在粒子物理标准模型中成功应用的量子化技术无法直接推广到引力领域: 广义相对论的微扰量子化会得到一个微扰层面不可重整的量子场论，在微扰论的每一阶都会出现新的无穷大 [1-4]。这催生了若干或多或少偏离连续量子场论基本原理的物理理论；近期的非专业介绍可参见 [5, 6] 。

The gravitational asymptotic safety program is one particular line of quantum gravity research. The program is conservative in the sense that it strives for a consistent and predictive theory of the gravitational interactions within the framework of quantum field theory by seeking a non-perturbative high-energy completion. Its core assumptions are that the gravitational degrees of freedom are encoded in the spacetime metric also at trans-Planckian scales. Moreover, the theory retains invariance under coordinate transformations. This assumption distinguishes the gravitational asymptotic safety program from Hořava-Lifshitz gravity [7] where this symmetry requirement is reduced to foliation-preserving diffeomorphisms, see [8] for a pedagogical discussion. The asymptotic safety hypothesis then stipulates that

引力渐近安全方案是量子引力研究的一个特定方向。该方案属于保守路线，它致力于在量子场论框架内得到一个自洽且具备预言能力的引力相互作用理论，为此寻找一种非微扰的高能完备化。其核心假设是: 即使在跨普朗克能标下，引力的自由度依然由时空度规编码，此外理论保留坐标变换下的不变性。这一假设将引力渐近安全方案与霍拉瓦-利夫希茨引力 [7] 区分开，后者将该对称性要求弱化至保叶状结构微分同胚，科普性讨论参见 [8]。渐近安全假设提出:

1. These ingredients give rise to an interacting renormalization group fixed point - called the Reuter fixed point.

1. 上述要素会产生一个相互作用的重整化群不动点——称为罗伊特不动点。

2. This fixed point controls the gravitational dynamics at trans-Planckian scales.

2. 该不动点控制跨普朗克能标下的引力动力学。

From a phenomenological perspective one also requires that the renormalization group flow emanating from the Reuter fixed point connects to a low-energy regime where the dynamics matches the one of general relativity to a good approximation.

从唯象视角来看，还要求起源自罗伊特不动点的重整化群流能够连接到低能区，且低能区的动力学可以很好地近似为广义相对论的动力学。

We stress that the central element of the gravitational asymptotic safety program - the existence of the Reuter fixed point coming with suitable properties - is not an input. It must be established based on first-principle computations. At the technical level, this requires tools applicable to the quantum field theory beyond the realm of the perturbation theory. This is a highly nontrivial endeavor. It took about 20 years from Weinberg's first formulation of the asymptotic safety hypothesis [9, 10] to the advent of renormalization group techniques which could be used to investigate this hypothesis in a systematic way [11].

我们要强调，引力渐近安全方案的核心——具备合适性质的罗伊特不动点的存在性——并非预先给定的输入，它必须通过第一性原理计算来确立。在技术层面，这需要适用于微扰论范围之外量子场论的研究工具，这是一项高度非平凡的工作。从温伯格首次提出渐近安全假设 [9, 10]，到能够系统研究该假设的重整化群技术出现 [11]，中间历时约 20 年。

Nowadays, there are two complementary computational approaches which naturally lend themselves to the exploration of the asymptotic safety mechanism in the context of gravity. Causal dynamical triangulations [12, 13] and Euclidean dynamical triangulations [14-18] use Monte Carlo techniques to investigate the phase space of quantum geometries resulting from the gravitational path integral. In this setting, the Reuter fixed point may manifest itself as a second-order phase transition [19] which allows to take the continuum limit in a controlled way. Alternatively, the Reuter fixed point can manifest itself in (approximate) solutions of the Wetterich equation [20, 21] .

如今，有两种互补的计算方法非常适合探索引力背景下的渐近安全机制。因果动态三角剖分 [12, 13] 与欧几里得动态三角剖分 [14-18] 使用蒙特卡洛技术，研究引力路径积分导出的量子几何相空间。在这一框架中，罗伊特不动点可以二级相变的形式呈现 [19]，从而允许我们以可控方式取连续极限。此外，罗伊特不动点也可以在维特里希方程 [20, 21] 的 (近似) 解中呈现。

This chapter will provide a basic introduction to the ideas underlying the gravitational asymptotic safety program (Section "The Asymptotic Safety Mechanism") before introducing the Wetterich equation [20,21] and its adaptation to gravity [11] as one of the main computational tools in the program (Section "The Functional Renormalization Group"). Section "The Einstein-Hilbert Truncation" illustrates how this tool is used in practical computations by working out the example of the Einstein-Hilbert truncation in a modern, background-independent way. Section "Concluding Comments" provides our conclusion and some brief comments on renormalization group techniques implemented by other approaches to quantum gravity.

本章将首先介绍引力渐近安全方案背后的基本思想 (对应“渐近安全机制”一节)，随后介绍作为该方案核心计算工具之一的维特里希方程 [20,21] 及其对引力的适配 [11](对应“泛函重整化群”一节)。“爱因斯坦-希尔伯特截断”一节通过实例说明，如何以现代的背景无关方式应用该工具进行实际计算，即爱因斯坦-希尔伯特截断。“结语”部分给出我们的结论，并简要介绍其他量子引力研究路线中使用的重整化群技术。

We stress that the exposition in this chapter is necessarily incomplete since it seeks to provide a concise introduction to the gravitational asymptotic safety program and the functional renormalization group which is accessible to a broader quantum gravity audience. For further details the reader is invited to consult the text books [22,23], lecture notes [24,25], and general reviews [26-28]. General introductions to the functional renormalization group are provided in [29-32] and there are topical reviews focusing on asymptotic safety in the presence of matter fields [33], the fluctuation approach to asymptotic safety [34], and its applications in the context of black holes [35] and cosmology [36]. Open issues have been discussed in the community report [37].

我们需要强调，本章的阐述必然不够完整，因为其旨在为更广泛的量子引力研究者提供引力渐近安全方案与泛函重整化群的简明入门介绍。读者若需进一步了解详情，可参考教科书 [22,23]、讲义 [24,25] 以及综述文献 [26-28]。泛函重整化群的基础介绍可见 [29-32]，此外还有专题综述分别关注包含物质场的渐近安全 [33]、渐近安全的涨落方法 [34]，以及该方案在黑洞 [35] 和宇宙学 [36] 中的应用。学界已在社群报告 [37] 中讨论了开放问题。

The Asymptotic Safety Mechanism

渐近安全机制

The insight that gravity could be asymptotically safe dates back to the seminal work of Weinberg [9, 10]. This initial proposal advocated asymptotic safety as a mechanism which renders physical scattering amplitudes finite (but nonvanishing) at energy scales exceeding the Planck scale. Motivated by computations showing that gravity in $d = 2 + \varepsilon$ spacetime dimensions possesses a nontrivial renormalization group (RG) fixed point [38,39], it was suggested that this family of fixed points admits an analytic continuation up to $d = 4$ where the corresponding fixed point should provide the high-energy completion of the gravitational interactions. The link between scattering amplitudes being finite and the RG fixed point builds on the insight that at such a fixed point all dimensionless quantities remain finite. If the fixed point controls the high-energy behavior, this property will also carry over to scattering amplitudes, which by themselves are dimensionless objects. This heuristic argument implies that it is not necessary that all dimensionless couplings remain finite. It suffices that the subset of couplings entering into physical observables (called essential couplings) attain their fixed-point values, as this is sufficient to ensure that the observables are well-behaved. A more detailed analysis of this scenario within the amplitude approach to asymptotic safety [40-42] revealed that there must be intricate relations between couplings and propagators. Most likely, these arise as a consequence of quantum scale symmetry realized at the fixed point [43].

引力可以满足渐近安全的洞见可追溯到温伯格的开创性工作 [9, 10]。这一最初提议将渐近安全倡导为一种机制: 在超过普朗克的能量标度下，它能让物理散射振幅保持有限 (但非零)。受计算结果表明 $d = 2 + \varepsilon$ 维时空的引力拥有非平庸重整化群 (RG) 不动点 [38,39] 的启发，有观点提出这类不动点可以解析延拓至 $d = 4$，对应的不动点将为引力相互作用提供高能完备化。散射振幅有限与 RG 不动点之间的关联建立在以下洞见之上: 在该不动点处，所有无量纲量都保持有限。如果不动点支配高能行为，这个性质也会延伸到散射振幅，而散射振幅本身就是无量纲对象。这一直观论证并不要求所有无量纲耦合都保持有限，只需进入物理可观测量的耦合子集 (称为本质耦合) 取到它们的不动点值即可，这足以保证可观测量表现良好。在渐近安全的振幅框架下对这一图景的更详细分析 [40-42] 表明，耦合与传播子之间必定存在复杂关联。这些关联极有可能源于不动点处实现的量子标度对称性 [43]。

The starting point for developing the idea of asymptotic safety is the functional integral over all Euclidean metrics,

发展渐近安全思想的起点是对所有欧几里得度量的泛函积分,

$$Z = \int \mathcal{D}h e^{-S[h]} \tag{1}$$

which would allow to determine all physical quantities of interest. In this respect, asymptotic safety shares the same starting point as Monte Carlo approaches to quantum gravity, foremost the causal dynamical triangulation [12,13] and Euclidean dynamical triangulation [14-18] programs as well as quantum Regge calculus [44,45].

它可以用来确定所有我们关心的物理量。在这一点上，渐近安全与量子引力的蒙特卡洛方法出发点相同，其中最主要的就是因果动态三角剖分 [12,13] 方案、欧几里得动态三角剖分 [14-18] 方案，以及量子里奇微积分 [44,45]。

The functional renormalization group then recasts the problem of performing this functional integral into the problem of solving a functional differential equation, the Wetterich equation for the effective average action Γ_k [11, 20, 21, 46] (derived in section ”The Functional Renormalization Group”):

泛函重整化群进而将完成该泛函积分的问题重构为求解一个泛函微分方程的问题，即有效平均作用量 Γ_k [11, 20, 21, 46] 的韦特里希方程 (在“泛函重整化群”小节推导):

$$k\partial_k \Gamma_k = \frac{1}{2} \mathrm{Tr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} k\partial_k \mathcal{R}_k\right]. \tag{2}$$

Here k is the coarse-graining scale and the trace contains an integration over loop momenta. The Wetterich equation implements the Wilsonian picture of renormalization in the following way: The regulator $\mathcal{R}_k$ appearing on the right-hand side separates the fluctuations into low- and high-momentum modes with respect to k .

此处 k 是粗粒化标度，迹中包含对圈动量的积分。韦特里希方程按如下方式实现了威尔逊的重整化图景: 出现在右侧的调节器 $\mathcal{R}_k$ 相对于 k 将涨落分离为低动量模和高动量模。

The change of Γ_k is then governed by integrating out quantum fluctuations with momenta $p^2 \approx k^2$. In this way, one arrives at a formulation that is much better behaved as the initial problem of solving the functional integral (1) in one stroke.

Γ_k 的变化由积分掉动量为 $p^2 \approx k^2$ 的量子涨落所支配。通过这种方式，我们得到了一个比一次性求解泛函积分 (1) 这个初始问题表现好得多的公式体系。

By construction, the propagators and vertices in the effective average action Γ_k include the quantum corrections due to the high-momentum fluctuations. In this sense, it provides an effective description of physics at length scales $l \sim k^{-1}$. This makes Γ_k a quite complicated object. Its natural habitat is the theory space $\mathcal{T}$, illustrated in Fig. 1. By definition, this space consists of all action functionals $A[\cdot]$ which can be constructed from the field content of the theory and meets its symmetry requirements. In the context of gravity, where the field content is given by (Euclidean) spacetime metrics $g_{\mu\nu}$, prototypical examples for these building blocks include the terms appearing in the Einstein-Hilbert action,

按构造，有效平均作用量 Γ_k 中的传播子和顶点已经包含了高动量涨落带来的量子修正。从这个意义上说，它提供了长度标度 $l \sim k^{-1}$ 下物理的有效描述。这使得 Γ_k 成为一个相当复杂的对象。它的自然生存空间是理论空间 $\mathcal{T}$，如图 1 所示。根据定义，这个空间由所有可由理论的场分量构造、且满足理论对称性要求的作用量泛函 $A[\cdot]$ 构成。在引力语境下，场分量由 (欧几里得) 时空度量 $g_{\mu\nu}$ 给出，这些构造块的典型例子包括爱因斯坦-希尔伯特作用量中出现的项，

$$\mathcal{O}_1 = \int d^d x \sqrt{g}, \ \mathcal{O}_2 = \int d^d x \sqrt{g} R, \tag{3}$$

where $\sqrt{g} \equiv \sqrt{\det(g)}$ and R is the Ricci scalar constructed from $g_{\mu\nu}$ (also see Table 1 for further examples). Given a basis $\{\mathcal{O}_i\}$ for these monomials, the effective average action can be expanded in this basis

其中 $\sqrt{g} \equiv \sqrt{\det(g)}$，$R$ 是由 $g_{\mu\nu}$ 构造的里奇标量 (更多示例参见表 1)。给定这些单项式的一组基 $\{\mathcal{O}_i\}$，有效平均作用量就可以在这组基下展开

$$\Gamma_k = \sum_i \bar{u}^i(k)\, \mathcal{O}_i \tag{4}$$

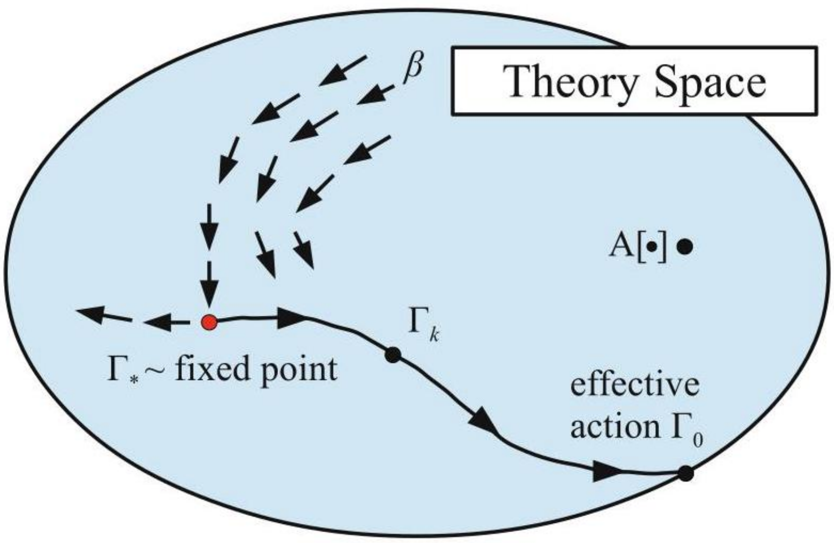


Fig. 1 Illustration of theory space and its structures: by definition, the theory space contains all action functionals $A[\cdot]$ which can be constructed from a given field content and obey the desired symmetries. The theory space comes with a vector field, the beta functions β . The integral curves of this vector field (RG trajectories) are exemplified by the black solid curve. The example emanates from a fixed point (red dot) with one UV-attractive ($\operatorname{Re}\theta_I > 0$) and one UV-repulsive ($\operatorname{Re}\theta_I < 0$) eigendirection. The endpoint of the RG trajectory at $k = 0$ coincides with the effective action Γ . Conventionally, all arrows point towards a lower coarse-graining scale, i.e., in the direction of integrating out fluctuation modes

图 1 理论空间及其结构示意图: 根据定义，理论空间包含所有可由给定场内容构造且满足要求对称性的作用量泛函 $A[\cdot]$ 。理论空间自带一个矢量场，即 β 函数 β 。该矢量场的积分曲线 (重整化群轨迹) 以黑色实线为例。示例轨迹从一个不动点 (红点) 出发，该不动点具有 1 个紫外吸引 ($\operatorname{Re}\theta_I > 0$) 本征方向和 1 个紫外排斥 ($\operatorname{Re}\theta_I < 0$) 本征方向。重整化群轨迹在 $k = 0$ 处的端点与有效作用量 Γ 重合。按照惯例，所有箭头都指向粗粒化尺度降低的方向，即积出涨落模的方向

Table 1 Illustration of the interaction monomials $\widetilde{\mathcal{O}}_l^n[g]$ appearing in the derivative expansion of $\overline{\Gamma}_k[g]$ at order n using the Weyl basis (70). The terms listed in the middle contribute to the graviton propagator in a four-dimensional flat background. Terms in the rightmost block contribute terms proportional to the background curvature in $\Gamma_k^{(2)}[h = 0; \bar{g}]$ and may be interpreted as "potential terms." Furthermore, $E = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ denotes the integrand of the Gauss-Bonnet term, which is topological in $d = 4$

表 1 使用外尔基 (70) 对 $\overline{\Gamma}_k[g]$ n 阶导数展开中出现的相互作用单项式 $\widetilde{\mathcal{O}}_l^n[g]$ 的说明。中间列出的项对四维平坦背景下的引力子传播子有贡献。最右块中的项贡献与 $\Gamma_k^{(2)}[h = 0; \bar{g}]$ 中背景曲率成正比的项，可解释为“势能项”。此外，$E = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ 表示高斯-博内项的被积函数，该项在 $d = 4$ 中是拓扑项

l / n	1	2	3	4	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
8	$R\Delta^2 R$	$C_{\mu\nu\rho\sigma}\Delta^2 C^{\mu\nu\rho\sigma}$	R^4	$R^2 R_{\mu\nu}R^{\mu\nu}$	...
6	$R\Delta R$	$C_{\mu\nu\rho\sigma}\Delta C^{\mu\nu\rho\sigma}$	R^3	$RR_{\mu\nu}R^{\mu\nu}$	+6 more
4	R^2	$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$	E		
2	R				
0	1				

The dependence on the coarse-graining scale is captured by the dimensionful couplings $\bar{u}^i(k)$. For the purpose of studying RG flows, it is useful to trade these dimensionful couplings with their dimensionless counterparts obtained by rescaling with k ,

对粗粒化尺度的依赖由量纲耦合 $\bar{u}^i(k)$ 刻画。研究重整化群流时，将这些量纲耦合替换为通过 k 标度得到的无量纲对应量是很有用的,

$$u^i(k) \equiv \bar{u}^i(k)\, k^{-d_i}, \tag{5}$$

where $d_i \equiv [\bar{u}^i]$ is the mass dimension of the coupling. The couplings u^i then serve as coordinates on $\mathcal{T}$.

其中 $d_i \equiv [\bar{u}^i]$ 是耦合的质量量纲。耦合 u^i 随后可作为 $\mathcal{T}$ 上的坐标。

Evaluating (2) for expansion (4) gives the component form of the functional renormalization group equation

对展开式 (4) 应用式 (2)，可得到泛函重整化群方程的分量形式

$$k\partial_k u^i(k) = \beta^i\left(\{u^j\}\right). \tag{6}$$

The beta functions $\beta^i\left(\{u^j\}\right)$ capture the dependence of the dimensionless couplings on the coarse-graining scale. Dimensional analysis entails that the functions $\beta^i\left(\{u^j\}\right)$ are independent of k , since this is the only dimensionful object in the construction. Thus, Eq. (6) constitutes an infinite-dimensional system of coupled, autonomous, first-order differential equations. Its solutions are called RG trajectories. The problem of performing the functional integral (1) is then translated into finding globally well-defined RG trajectories

β 函数 $\beta^i\left(\{u^j\}\right)$ 刻画了无量纲耦合对粗粒化尺度的依赖。量纲分析可得，函数 $\beta^i\left(\{u^j\}\right)$ 与 k 无关，因为 k 是该构造中唯一的量纲对象。因此，式 (6) 构成了无穷维耦合自治一阶微分方程组，其解称为重整化群轨迹。求解泛函积分 (1) 的问题由此转化为寻找全局良定的重整化群轨迹

$$k \to \Gamma_k, \ k \in [0, \infty] \tag{7}$$

which exist for all values of the coarse-graining scale k .

这类轨迹对所有粗粒化尺度 k 都存在。

By definition, RG fixed points $\left\{u_*^j\right\}$ are stationary points of the system (6), satisfying

根据定义，重整化群不动点 $\left\{u_*^j\right\}$ 是方程组 (6) 的平稳点，满足

$$\beta^i\left(\left\{u_*^j\right\}\right) = 0, \ \forall i. \tag{8}$$

As a consequence, it takes infinite amount of ”RG time” for an RG trajectory to actually reach the fixed point. In this way fixed points can provide a well-defined limit $k \to \infty$ in which all dimensionless couplings $u^i(k) \to u_*^i$ remain finite. Thus, fixed points are natural candidates for providing a well-defined high-energy completion of a theory. It is this concept that underlies the Wilsonian picture of renormalization.

因此，一条重整化群轨迹需要无穷多的“重整化群时间”才能真正到达不动点。据此，不动点可以提供一个所有无量纲耦合 $u^i(k) \to u_*^i$ 都保持有限的良定极限 $k \to \infty$ 。因此，不动点是构造理论良定高能完备化的自然候选。这就是威尔逊重整化图景背后的核心概念。

At this point it is interesting to inquire about the conditions for an RG trajectory being dragged into a fixed point as $k \to \infty$. This question is closely related to the predictive power of the construction. In the vicinity of a fixed point $\{u_*^i\}$, the properties of the RG flow can be studied by linearizing the system (6),

在此处，探究当 $k\to\infty$ 时 RG 轨迹被吸引到不动点的条件是很有意义的。这个问题与该构造的预言能力密切相关。在不动点 $\{u_*^i\}$ 附近，可以通过线性化系统 (6) 来研究 RG 流的性质，

$$k\partial_k u^i(k) = \sum_j B^i{}_j \left(u^j(k) - u_*^j\right) + O\left(u^2\right). \tag{9}$$

Here

此处

$$B^i{}_j \equiv \frac{\partial}{\partial u^j}\beta^i\Big|_{u=u^*} \tag{10}$$

is the stability matrix associated with the fixed point. The solutions of (9) are readily given in terms of the right eigenvectors V_I and stability coefficients θ_I of B ,

是对应该不动点的稳定性矩阵。方程 (9) 的解可以很容易地用右本征向量 V_I 和 B 的稳定性系数 θ_I 表示，

$$\sum_j B^i{}_j V_I^j = -\theta_I V_I^i,\ \forall I, \tag{11}$$

and take the form

形式如下

$$u^i(k) = u_*^i + \sum_J C_J V_J^i \left(\frac{k_0}{k}\right)^{\theta_J}. \tag{12}$$

Here C_J are constants of integration and k_0 denotes an arbitrary reference scale.

此处 C_J 是积分常数，k_0 表示任意参考标度。

Inspecting (12) reveals that eigendirections with $\mathrm{Re}(\theta_I) > 0$ are attracted by the fixed point as $k\to\infty$ while the ones with $\mathrm{Re}(\theta_I) < 0$ are repulsive in this limit. The corresponding scaling operators are called ”UV-relevant” and ”UV-irrelevant”, respectively. This suggests splitting the set $\{C_I\}$ according to

观察式 (12) 可以发现，当 $k\to\infty$ 时，本征方向中满足 $\mathrm{Re}(\theta_I) > 0$ 的会被不动点吸引，而满足 $\mathrm{Re}(\theta_I) < 0$ 的本征方向在此极限下是排斥的。对应的标度算子分别被称为“UV 相关”算子和“UV 无关”算子。据此我们可以将集合 $\{C_I\}$ 拆分为

$$\left\{C_I^{\text{relevant}}\right\} = \{C_I \mid \mathrm{Re}(\theta_I) > 0\},\ \left\{C_I^{\text{irrelevant}}\right\} = \{C_I \mid \mathrm{Re}(\theta_I) < 0\}. \tag{13}$$

The case $\mathrm{Re}\,\theta_I = 0$ corresponds to a marginal direction. Determining whether this direction is UV-attractive or UV-repulsive requires going beyond the linear approximation (12) and will not be discussed in detail here.

Re $\theta_I = 0$ 的情况对应临界方向。要确定该方向是 UV 吸引还是 UV 排斥，需要超出线性近似 (12) 的范围，本文不对此展开详细讨论。

The condition that the fixed point controls the UV behavior of the RG trajectory then enforces $C_I^{\text{irrelevant}} = 0$, for all I . The solutions meeting this condition span the UV-critical hypersurface of the fixed point. The $\{C_I^{\text{relevant}}\}$ are the free parameters of the construction and label the solutions within this hypersurface. Their value is unconstrained by demanding a well-defined UV-completion and must be determined by other theoretical considerations or experimental input. This discussion also shows that fixed points with a lower-dimensional UV-critical hypersurface have a higher predictive power.

不动点控制 RG 轨迹的 UV 行为这一条件，要求对所有 I 都满足 $C_I^{\text{irrelevant}} = 0$ 。满足该条件的解构成不动点的 UV 临界超曲面。$\{C_I^{\text{relevant}}\}$ 是该构造的自由参数，标记了这个超曲面上的不同解。它们的取值不受“紫外完备性良好”这一要求的约束，必须通过其他理论考量或实验输入来确定。该讨论也表明，UV 临界超曲面维度更低的不动点具备更高的预言能力。

Up to this point, our discussion of a high-energy completion referred to a generic renormalization fixed point. It is then customary to distinguish among a Gaussian fixed point (GFP) and a non-Gaussian fixed point (NGFP). The definition of the former is that the critical exponents of its stability matrix agree with the canonical mass dimension of the corresponding coupling $\theta_I = d_I$. This signals that the underlying theory is the free theory. At a NGFP, the stability coefficients receive quantum corrections,

到这里为止，我们对高能完备化的讨论都是针对一般重整化不动点展开的。通常我们会将其区分为高斯不动点 (GFP) 和非高斯不动点 (NGFP)。前者的定义是: 其稳定性矩阵的临界指数与对应耦合 $\theta_I = d_I$ 的正则质量维数一致。这表明对应的基础理论是自由理论。在非高斯不动点处，稳定性系数会获得量子修正，

$$\theta_I = d_I + \text{ quantum corrections.} \tag{14}$$

The latter indicate that the theory linked to the fixed point is interacting. Notably, this definition of a Gaussian and non-Gaussian fixed point is not based on the values $\{u_*^i\}$. Since the spectrum of the stability matrix is invariant under a redefinition $u^i \mapsto \tilde{u}^i(u^i)$, this characterization is independent of a specific choice of ”coordinate system” on $\mathcal{T}$. An important subset of NGFPs are ”almost-Gaussian” NGFPs. In this case the quantum corrections in (14) are weak in the sense that the θ_I are dominated by their classical part. This implies that classical power counting is still a valid guiding principle for determining whether a scaling operator is relevant or irrelevant. Beyond the class of ”almost Gaussian” NGFPs, there could also be fixed points where the critical exponents are dominated by quantum effects. The systematic investigation of this possibility is beyond the scope of most of current searches for RG fixed points based on functional renormalization group equations though. Some insights on potential stability patterns associated with such fixed points have recently been discussed based on the composite operator equation [47, 48], indicating that studying such fixed points requires approximations at a significant level of complexity as well as dedicated search strategies. Depending on whether the high-energy completion is provided by a GFP or a NGFP, the theory is termed ”asymptotically free” or ”asymptotically safe.” A prototypical example of the former case is quantum chromodynamics, while the latter case is realized by gravity in $d = 2 + \varepsilon$ spacetime dimensions [38,39].

量子修正表明与该不动点关联的理论是相互作用理论。值得注意的是，我们对高斯不动点和非高斯不动点的定义并不基于取值 $\{u_*^i\}$。由于稳定性矩阵的谱在 $u^i \mapsto \bar{u}^i(u^i)$ 重新定义下保持不变，因此该刻画不依赖于 $\mathcal{T}$ 上“坐标系”的特定选择。非高斯不动点的一个重要子类是“近高斯”非高斯不动点。这类情况中，式 (14) 里的量子修正很弱，θ_I 由其经典部分主导。这意味着经典幂计数依然可以作为判断标度算子相关或无关的有效指导原则。而在“近高斯”非高斯不动点这类之外，还可能存在临界指数由量子效应主导的不动点。不过，对这种可能性的系统研究超出了当前基于泛函重整化群方程搜寻 RG 不动点大多数工作的范围。最近已有研究基于复合算子方程讨论了这类不动点可能的稳定性模式 [47,48]，研究表明探究这类不动点需要相当复杂的近似以及专门的搜寻策略。根据高能完备性由高斯不动点还是非高斯不动点给出，对应的理论分别被称为“渐近自由”理论和“渐近安全”理论。前者的典型例子是量子色动力学，后者则由 $d = 2 + \varepsilon$ 维时空的引力实现 [38,39]。

We conclude this section with two clarifications. For a globally well-defined RG trajectory, the solutions (7) interpolate between the microscopic dynamics determined by the RG fixed point for $k \to \infty$ and the standard effective action $\lim_{k\to 0} \Gamma_k = \Gamma$. All physics should then be extracted from Γ using its quantum corrected propagators and vertices. Similarly to (4), Γ can be expanded in a basis of the theory space

我们在本节最后做两点说明。对于全局良定义的 RG 轨迹，方程 (7) 的解会插值于 $k \to \infty$ 处由 RG 不动点决定的微观动力学与标准有效作用量 $\lim_{k\to 0} \Gamma_k = \Gamma$ 之间。随后所有物理结果都应当从 Γ 中，借助其量子修正传播子和顶点提取得到。与式 (4) 类似，Γ 可在理论空间的一组基下展开

$$\Gamma = \sum_i \bar{u}^i_{\text{eff}} \, \mathcal{O}_i \tag{15}$$

with the relation between the couplings being $\bar{u}^i_{\text{eff}} = \lim_{k\to 0} \bar{u}^i(k)$. This expansion is similar to the one encountered in effective field theory where the $\mathcal{O}_i$ are organized according to their canonical mass dimension and the sum is truncated at a given order. The key difference to the effective field theory approach is that the RG flow determines the effective couplings in terms of the free coefficients $\{C_I^{\text{relevant}}\}$:

其中耦合间的关系为 $\bar{u}^i_{\text{eff}} = \lim_{k\to 0} \bar{u}^i(k)$。该展开与有效场论中常用的展开类似，有效场论中 $\mathcal{O}_i$ 按正则质量维度排序，求和会在给定阶数处截断。它与有效场论方法的关键区别在于，RG 流会以自由系数 $\{C_I^{\text{relevant}}$ 的形式确定有效耦合}:

$$\bar{u}^i_{\text{eff}} = \bar{u}^i_{\text{eff}}\left(C_I^{\text{relevant}}\right). \tag{16}$$

Provided that there are more couplings $\bar{u}^i_{\text{eff}}$ than free parameters C_I^{relevant}, the high-energy completion induces (a potentially infinite number of) relations between the effective couplings. These provide predictions which can be confronted with theoretical consistency requirements and experimental data. On this basis one can deduce whether a given RG fixed point leads to low-energy physics compatible with nature. This also allows to falsify the construction, provided that the properties of the fixed point and its UV-critical surface are known at a sufficient level of detail.

若耦合 $\bar{u}^i_{\text{eff}}$ 的数量多于自由参数 C_I^{relevant} ，高能完备化会诱导出有效耦合之间 (可能无穷多组) 的关系。这些关系给出的预言可以与理论自洽性要求和实验数据比对。在此基础上我们可以推断给定 RG 不动点是否能产生与自然相容的低能物理。如果不动点及其紫外临界面的性质已经被了解得足够详细，这种构造也可以被证伪。

We also stress that the dependence of couplings on the coarse-graining scale k should not be identified with the running of a coupling with respect to a physical energy scale; see [37, 49] for instructive examples. Generically, the couplings appearing in (16) are not constant but come in the form of form factors depending on the momenta of the fields in a nontrivial way. In the simplest case (cf. (72)), this dependence contains a single momentum scale

我们还要强调，不能将耦合对粗粒化标度 k 的依赖等同于耦合随物理能标的跑动；关于启发性的例子可见文献 [37,49]。一般来说，式 (16) 中的耦合不是常数，而是以形状因子的形式存在，非平凡地依赖于场的动量。最简单的情况 (参见式 (72)) 中，这种依赖只包含一个动量标度

$$\bar{u}^i_{\text{eff}} \to \bar{u}^i_{\text{eff}}\left(p^2\right). \tag{17}$$

In practice, the value of the coupling is then measured at a fixed momentum scale μ^2 . The nontrivial p -dependence then induces the "running" of the coupling with respect to its value determined at the reference scale. In the simplest case, this is the logarithmic running of a dimensionless coupling seen in perturbation theory, but the momentum dependence can be significantly more involved than that.

实践中，耦合的取值会在固定动量标度 μ^2 处测量。非平凡的 p 依赖会进一步诱导耦合相对于参考标度处取值的“跑动”。最简单的情况就是微扰论中无量纲耦合的对数跑动，但动量依赖可以比这复杂得多。

The Functional Renormalization Group

泛函重整化群

The basic idea of a functional renormalization group equation (FRGE) is to recast the functional integral over quantum fluctuations in terms of a functional differential equation. The FRGE implements Wilson's modern viewpoint on renormalization [50]: in contrast to a perturbative approach based on evaluating Feynman diagrams, quantum fluctuations are not integrated over in one stroke. Instead, they are integrated out "shell-by-shell" in momentum space starting with the most energetic ones. This leads to a one-parameter family of effective actions Γ_k whose propagators and vertices already contain the quantum corrections from fluctuations with momenta $p^2 \gtrsim k^2$. The textbook effective action Γ is recovered in the limit where all fluctuations are integrated out, $\Gamma = \lim_{k\to 0} \Gamma_k$.

泛函重整化群方程 (FRGE) 的核心思想是将量子涨落的泛函积分重新表述为泛函微分方程。FRGE 践行了威尔逊关于重整化的现代观点 [50]: 与基于计算费曼图的微扰方法不同，量子涨落并非一步完成积分，而是从能量最高的涨落开始，在动量空间中“逐壳”积出。这得到了单参数族的有效作用量 Γ_k ，其传播子和顶点已经包含了动量为 $p^2 \gtrsim k^2$ 的涨落带来的量子修正。当所有涨落都被积出，即 $\Gamma = \lim_{k\to 0} \Gamma_k$ 时，就得到了标准教材中的有效作用量 Γ 。

The FRGE most frequently used in hands-on computations is the Wetterich equation [11, 20, 21, 51] . This section reviews its construction for scalar fields (Section ”The Wetterich Equation for Scalar Field Theory”) before extending the formalism to gravity (Section ”The Wetterich Equation for Gravity”). The most common non-perturbative approximation techniques to this equation are introduced in section ”Common Approximation Schemes” and important extensions giving structural insights to the gravitational renormalization group flow are summarized in section ”Further Developments”.

实际计算中最常用的 FRGE 是韦特里希方程 [11, 20, 21, 51] 。本节首先介绍它对标量场的构造 (章节“标量场论的韦特里希方程”)，再将形式化方法推广到引力 (章节“引力的韦特里希方程”。我们将在“常用近似方案”章节介绍该方程最常见的非微扰近似技巧，并在“进一步发展”章节总结对引力重整化群流给出结构洞见的重要扩展。

The Wetterich Equation for Scalar Field Theory

标量场论的韦特里希方程

The Wetterich equation is a universal tool for studying the RG flow of theories built from essentially any field content [32]. In order to introduce this tool with the absolute minimum of technicalities, we first focus on a real scalar field φ living on a d -dimensional Euclidean spacetime $\left(\mathbb{R}^d, \delta_{\mu\nu}\right)$. For pedagogical reasons, we first review the construction of the effective action in this setting before introducing the effective average action and its FRGE.

韦特里希方程是研究由任意场内容构成的理论的重整化群流的通用工具 [32]。为了用最少的技术细节引入这一工具，我们首先聚焦于定义在 d 维欧几里得时空 $\left(\mathbb{R}^d, \delta_{\mu\nu}\right)$ 上的实标量场 φ 。出于教学目的，我们先回顾该框架下有效作用量的构造，再引入有效平均作用量及其泛函重整化群方程。

We start from the generating functional of correlation functions (path integral)

我们从关联函数的生成泛函 (路径积分) 开始

$$Z[J] \equiv \frac{1}{N} \int \mathcal{D}\varphi \exp\left\{-S[\varphi] + \int d^d x J(x) \varphi(x)\right\}. \tag{18}$$

Here $N \equiv \int \mathcal{D}\varphi \exp\{-S[\varphi]\}$ is a normalization factor and $J(x)$ a source coupling to the quantum field. The dynamics of the field is governed by the bare action $S[\varphi]$ which is kept arbitrary at this point. Generically, this generating functional diverges and we implicitly assume that it has been suitably regularized by including a UV cutoff. Eq. (18) allows to construct expectation values of operators $\mathcal{O}$

式中 $N \equiv \int \mathcal{D}\varphi \exp\{-S[\varphi]\}$ 是归一化因子，$J(x)$ 是与量子场耦合的源。场的动力学由裸作用量 $S[\varphi]$ 支配，本文在此处保留其任意性。一般而言，该生成泛函是发散的，我们默认已经通过引入紫外截止对其做了适当正则化。式 (18) 可以用来构造算符 $\mathcal{O}$ 的期望

$$\langle \mathcal{O}[\varphi] \rangle \equiv \frac{1}{N} \int \mathcal{D}\varphi \mathcal{O}[\varphi] \exp\{-S[\varphi]\}. \tag{19}$$

In particular, expectation values of operators polynomial in φ can be obtained by taking functional derivatives with respect to the source and subsequently setting J to zero

特别地，对源取泛函导数再令 J 等于零，即可得到 φ 多项式算符的期望

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle = \left. \frac{\delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J=0}. \tag{20}$$

Here, the normalization factors are chosen such that $\langle \mathbb{1} \rangle = 1$. Based on the path integral (18), one obtains the functional $W[J]$ generating all connected Green’s functions by setting

此处我们选取归一化因子使得 $\langle \mathbb{1} \rangle = 1$ 成立。基于路径积分 (18)，通过如下定义可以得到生成所有连通格林函数的泛函 $W[J]$

$$Z[J] \equiv e^{W[J]} \tag{21}$$

We then introduce the mean field $\phi(x)$ as the expectation value of $\varphi(x)$:

随后我们引入平均场 $\phi(x)$ 作为 $\varphi(x)$ 的期望:

$$\phi(x) = \langle \varphi(x) \rangle = \frac{\delta W[J]}{\delta J(x)}. \tag{22}$$

Finally, one constructs the effective action $\Gamma[\phi]$ as the Legendre transform of $W[J]$. If the relation (22) can be solved for the source, giving $J[\phi]$, it takes the form

最后，我们构造有效作用量 $\Gamma[\phi]$ 作为 $W[J]$ 的勒让德变换。若式 (22) 可以反解出源 $J[\phi]$ ，其形式为

$$\Gamma[\phi] = \int d^d x J[\phi](x) \phi(x) - W[J[\phi]]. \tag{23}$$

In the general case, the effective action is obtained as the Legendre-Fenchel transform $\Gamma[\phi] = \sup_{J(x)} \big(\int d^d x J[\phi](x) \phi(x) - W$. In the sequel, formulas are understood to include the supremum if needed. The fact that $W[J]$ and $\Gamma[\phi]$ are related by a Legendre transform implies that

一般情况下，有效作用量由勒让德-芬切尔变换 $\Gamma[\phi] = \sup_{J(x)} \big(\int d^d x J[\phi](x) \phi(x) - W[J[\phi]] \big)$ 得到。在下文中，公式默认已在需要时包含上确界。$W[J]$ 与 $\Gamma[\phi]$ 由勒让德变换联系这一性质意味着

$$\int d^d y \frac{\delta^2 W[J]}{\delta J(x_1)\delta J(y)} \frac{\delta^2 \Gamma[\phi]}{\delta\phi(y)\delta\phi(x_2)} = \delta^d(x_1 - x_2). \tag{24}$$

The effective action provides the equation of motion for the mean field in the presence of a source,

有效作用量给出了存在源时平均场满足的运动方程,

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = J(x). \tag{25}$$

Higher-order functional derivatives generate the one-particle irreducible (1P) n -point functions

高阶泛函导数可以生成单粒子不可约 (1PI) n 点函数

$$\Gamma^{(n)}[\phi] \equiv \frac{\delta^n \Gamma[\phi]}{\delta\phi(x_1)\cdots\delta\phi(x_n)} = \langle\varphi(x_1)\cdots\varphi(x_n)\rangle_{1\text{PI}}. \tag{26}$$

Equation (24) then entails that the second functional derivative of $\Gamma[\phi]$ encodes the quantum corrected propagator

式 (24) 表明，$\Gamma[\phi]$ 的二阶泛函导数编码了量子修正传播子

$$\left(\Gamma^{(2)}(x_1, x_2)\right)^{-1} = W^{(2)}(x_1, x_2) = G(x_1, x_2). \tag{27}$$

Scattering processes are described by tree-level Feynman diagrams constructed from the propagators and vertices extracted from $\Gamma[\phi]$. In this sense, the effective action is the quantum analog of the classical action, since it encodes the quantum physics at tree level. Determining $\Gamma[\phi]$ is therefore often considered as equivalent to solving the quantum theory.

散射过程可以由从 $\Gamma[\phi]$ 提取的传播子和顶点构造的树阶费曼图描述。从这个意义上说，有效作用量是经典作用量的量子类比，它在树阶就编码了全部量子物理。因此，确定 $\Gamma[\phi]$ 通常被认为等价于求解该量子理论。

The construction of the effective average action $\Gamma_k[\phi]$ proceeds along very similar lines. The key modification occurs at the level of the generating functional (18) which is supplemented by an IR regulator

有效平均作用量 $\Gamma_k[\phi]$ 的构造遵循非常相似的思路。核心修改发生在生成泛函 (18) 层面: 我们为其补充了一个红外调节器

$$\Delta S_k[\varphi] = \frac{1}{2}\int d^d x \varphi(x) R_k\left(-\partial^2\right)\varphi(x). \tag{28}$$

The purpose of this extra ingredient is to provide a k -dependent mass term for quantum fluctuations with moments $p^2 \ll k^2$. In the simplest case, this is implemented by requiring that the regulator $R_k\left(p^2\right)$ satisfies

这个额外项的作用是为动量为 $p^2 \ll k^2$ 的量子涨落提供一个依赖于 k 的质量项。最简单的实现方式是要求正规化子 $R_k\left(p^2\right)$ 满足

$$R_k\left(p^2\right) \approx \begin{cases} k^2 & \text{for } p^2 \ll k^2, \\ 0 & \text{for } p^2 \gg k^2. \end{cases} \tag{29}$$

Examples of regulators used in practical computations include the (smooth) exponential cutoff,

实际计算中常用的正规化子包括 (光滑) 指数截断,

$$R_k\left(p^2\right) = p^2\left(\exp\left(p^2/k^2\right) - 1\right)^{-1}, \tag{30}$$

and Litim-type regulators,

以及 Litim 型正规化子,

$$R_k\left(p^2\right) = \left(k^2 - p^2\right)\Theta\left(1 - p^2/k^2\right), \tag{31}$$

where $\Theta(x)$ is the Heaviside step function. Adding (28) to the weight in the generating functional (18) induces a dependence on the scale k

其中 $\Theta(x)$ 是海维赛德阶跃函数。将式 (28) 加入生成泛函 (18) 的权重中，就会引入对能标 k 的依赖

$$Z_k[J] = \frac{1}{N}\int \mathcal{D}\varphi \exp\left\{-S[\varphi] - \Delta S_k[\varphi] + \int d^d x J(x)\varphi(x)\right\}. \tag{32}$$

The effect is that the contribution of modes with $p^2 \ll k^2$ to the generating functional becomes suppressed, while the modes with $p^2 \gg k^2$ are integrated out in the usual way. Thus, k acquires a natural interpretation as a coarse-graining scale, marking the scale up to which microscopic quantum fluctuations are included in the generating functional.

其作用是，$p^2 \ll k^2$ 模式对生成泛函的贡献被抑制，而 $p^2 \gg k^2$ 模式则按常规方式被积出。因此，k 可以自然地被诠释为粗粒化能标，它标记了微观量子涨落被纳入生成泛函的上限能标。

Following the steps leading to the effective action, we then define the (now k -dependent) generating functional for connected Green’s functions $W_k[J]$ by

遵循导出有效作用量的步骤，我们接下来定义依赖于 k 的连通格林函数生成泛函 $W_k[J]$，形式为

$$Z_k[J] = \exp\left[W_k[J]\right] \tag{33}$$

By definition, the effective average action is then given by a modified Legendre transform of $W_k[J]$:

根据定义，有效平均作用量由 $W_k[J]$ 的修正勒让德变换给出:

$$\Gamma_k[\phi] \equiv \int d^d x J[\phi](x)\phi(x) - W_k[J] - \Delta S_k[\phi]. \tag{34}$$

For $k = 0$ the IR regulator in the definition of $W_k[J]$ as well as $\Delta S_k[\phi]$ vanish and (34) agrees with the definition of the effective action (23):

当 $k=0$ 时，$W_k[J]$ 和 $\Delta S_k[\phi]$ 定义中的红外正规化子都消失，式 (34) 与有效作用量的定义 (23) 一致:

$$\lim_{k\to 0}\Gamma_k[\phi]=\Gamma[\phi] \tag{35}$$

The key virtue of the effective average action is that its k -dependence is governed by a functional renormalization group equation, the Wetterich equation. This equation is formally exact in the sense that no approximations are made in its derivation. The construction of the Wetterich equation then proceeds along the following lines. We start by introducing the RG time $t\equiv \ln k/k_0$, with k_0 being an arbitrary reference scale, so that $\partial_t = k\partial_k$. We then consider the auxiliary generating functional

有效平均作用量的核心优点是，它对 k 的依赖由一个泛函重整化群方程即 Wetterich 方程支配。该方程在推导过程中没有引入任何近似，因此是形式上精确的。Wetterich 方程的构造过程如下。我们首先引入重整化群时间 $t\equiv \ln k/k_0$ ，其中 k_0 是任意参考能标，因此有 $\partial_t = k\partial_k$ 。接下来我们考虑辅助生成泛函

$$\tilde{\Gamma}_k[\phi]\equiv\int d^dxJ[\phi](x)\phi(x)-W_k[J]\,. \tag{36}$$

Taking a partial derivative of this definition with respect to the RG time yields

对该定义关于 RG 时间取偏导可得

$$\partial_t\tilde{\Gamma}_k[\phi]=-\partial_tW_k[J]=\frac{1}{2}\int d^dx\int d^dy\langle\varphi(x)\varphi(y)\rangle\partial_tR_k(x,y)\,. \tag{37}$$

Here we have used that $\partial_tW_k[J]=\partial_t\ln Z_k[J]$ with

此处我们用到了 $\partial_tW_k[J]=\partial_t\ln Z_k[J]$ ，其中

$$\begin{aligned}\partial_t\ln Z_k[J]=&-\frac{1}{2Z_k}\int\mathcal{D}\varphi\int d^dx\int d^dy\varphi(x)\partial_tR_k(x,y)\varphi(y)\times\\&\times\exp\left\{-S[\varphi]-\Delta S_k[\varphi]+\int d^dxJ(x)\varphi(x)\right\}\\=&-\frac{1}{2}\int d^dx\int d^dy\langle\varphi(x)\varphi(y)\rangle\partial_tR_k(x,y)\,,\end{aligned} \tag{38}$$

in the second step. We then introduce the (k -dependent) mean field

在第二步得到该式。随后我们引入依赖于 k 的平均场

$$\phi(x)=\langle\varphi(x)\rangle=\frac{\delta W_k[J]}{\delta J(x)}, \tag{39}$$

together with the two-point functions

以及两点函数

$$\langle \varphi(x)\varphi(y)\rangle_{\mathrm{c}} \equiv \frac{\delta^2 W_k[J]}{\delta J(x)\,\delta J(y)},\ \tilde{\Gamma}_k^{(2)}(x,y) \equiv \frac{\delta^2 \tilde{\Gamma}_k[\phi]}{\delta\phi(x)\,\delta\phi(y)}. \tag{40}$$

Since $\tilde{\Gamma}_k[\phi]$ and $W_k[J]$ are again related by a Legendre transform, these functionals are again each other's inverse; cf. Eq. (24). This allows to express the two-point function appearing in the relation (37) in terms of $\tilde{\Gamma}_k^{(2)}(x,y)$

由于 $\tilde{\Gamma}_k[\phi]$ 和 $W_k[J]$ 仍由勒让德变换联系，这两个泛函互为逆；参见式 (24)。由此我们可以将式 (37) 中的两点函数用 $\tilde{\Gamma}_k^{(2)}(x,y)$ 表示

$$\langle\varphi(x)\varphi(y)\rangle = \langle\varphi(x)\varphi(y)\rangle_{\mathrm{c}} + \langle\varphi(x)\rangle\langle\varphi(y)\rangle$$

$$= \left(\tilde{\Gamma}_k^{(2)}(x,y)\right)^{-1} + \phi(x)\phi(y). \tag{41}$$

Here we used the definition of the (now k -dependent) mean field when recasting the last term. Substituting this relation into (37) then yields

此处我们在整理最后一项时用到了依赖于 k 的平均场的定义。将该关系代入式 (37) 后可得

$$\partial_t\tilde{\Gamma}_k = \frac{1}{2}\int d^dx\int d^dy\left[\left(\tilde{\Gamma}_k^{(2)}(x,y)\right)^{-1}\partial_t R_k(x,y)\right] + \partial_t\Delta S_k[\phi]. \tag{42}$$

Bringing the second term to the left-hand side and using that $\Gamma_k = \tilde{\Gamma}_k - \Delta S_k$ allows to rewrite this equation in terms of the effective average action

将第二项移到左侧，并利用 $\Gamma_k = \tilde{\Gamma}_k - \Delta S_k$ ，我们可以将该方程改写为用有效平均作用量表示的形式

$$\partial_t\Gamma_k[\phi] = \frac{1}{2}\int d^dx\int d^dy\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1}\partial_t R_k\right]. \tag{43}$$

Noticing that the integrals on the right-hand side actually correspond to taking the trace of the argument, we arrive at the Wetterich equation in its iconic form

注意到右侧的积分实际上对应于对自变量求迹，我们得到了标志性形式的 Wetterich 方程

$$\partial_t\Gamma_k = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1}\partial_t R_k\right]. \tag{44}$$

The Wetterich equation exhibits several remarkable features arising from the interplay of $R_k\left(p^2\right)$ in the numerator and denominator of the trace argument. In the propagator term $\left(\Gamma_k^{(2)} + R_k\right)^{-1}$, the regulator provides a mass to the fluctuations, ensuring the absence of IR singularities as long as k is finite. In the numerator, the condition $R_k\left(p^2\right) \to 0$ for $p^2 \gg k^2$ entails that the trace argument vanishes for high-momentum modes. As a consequence the right-hand side is IR and UV-finite and any UV regulator implicit in the definition of the initial functional integral can be removed trivially. In some practical computations, as, e.g., in the computation of spectral flows [52], one may want to resort to regulators $R_k\left(p^2\right)$ where this falloff property in the UV does not hold. In this case, the flow equation must be supplemented by additional counterterms absorbing the UV divergences.

Wetterich 方程具有若干显著特性，这些特性源于迹自变量分子与分母中 $R_k(p^2)$ 的相互作用。在传播子项 $\left(\Gamma_k^{(2)}+R_k\right)^{-1}$ 中，调节器给涨落赋予了质量，只要 k 有限，就保证不会出现红外奇点。在分子中，针对 $p^2 \gg k^2$ 的条件 $R_k(p^2) \to 0$ 意味着高能动量模式的迹自变量为零。因此，方程右侧对红外和紫外都是有限的，初始泛函积分定义中隐含的任何紫外调节器都可以轻易移除。在一些实际计算中，例如谱流的计算 [52]，可能需要使用不满足这种紫外衰减性质的调节器 $R_k(p^2)$ 。这种情况下，流方程必须补充额外的抵消项来吸收紫外发散。

The regulator structure furthermore entails that the trace argument is peaked at momenta $p^2 \approx k^2$. Hence, the flow of $\Gamma_k[\phi]$ is driven by integrating out quantum fluctuations whose momenta are comparable to the coarse-graining scale k . In this way the Wetterich equation implements the Wilsonian picture of renormalization, integrating out quantum fluctuations shell-by-shell in momentum space. Notably, Eq. (44) allows to start from any initial condition Γ_A and integrate its RG flow towards the infrared. Thus, the Wetterich equation does not require specifying a bare action a priori. These are obtained as the fixed points of the RG flow through the reconstruction problem [53].

此外，调节器结构还使得迹自变量在动量 $p^2 \approx k^2$ 处达到峰值。因此，$\Gamma_k[\phi]$ 的流由积分掉动量与粗粒化标度 k 相当的量子涨落驱动。以此方式，Wetterich 方程实现了威尔逊重整化图景，在动量空间中逐层积分掉量子涨落。值得注意的是，式 (44) 可以从任意初始条件 Γ_A 出发，向红外方向积分其重整化群流。因此，Wetterich 方程不需要预先指定裸作用量，裸作用量可以通过重构问题作为重整化群流的不动点得到 [53]。

We also observe that the combination of propagator and regulator within the trace induces a projective feature. Any k -independent rescaling of the fluctuation field affects the regulator and propagator in the same way, so that such rescalings drop out from the right-hand side of the equation. This renders the flow equation invariant with respect to certain classes of field redefinitions.

我们还发现，迹内传播子与调节器的组合带来了投影性质。涨落场的任何不依赖 k 的重标度会以相同方式影响调节器和传播子，因此这类重标度会从方程右侧消去。这使得流方程在 certain classes of field redefinitions 下保持不变。

The Wetterich Equation for Gravity

引力的韦特里希方程

In the previous section, we derived the Wetterich equation (44) for a scalar field theory with a single degree of freedom. Its extension to gauge fields and fermions is conceptually straightforward. In the context of gravity the construction faces two conceptual obstacles though. Firstly, our understanding of classical gravity based on general relativity indicates that gravitational interactions are mediated through the curvature of spacetime. This implies that spacetime itself becomes a dynamical and, in the context of the quantum theory, also fluctuating object. Hence, the concept of a fixed, non-dynamical spacetime providing the stage for the dynamics is lost at this point. This raises the question about how to define the coarse-graining scale

k . Secondly, gravity shares some properties of a gauge theory. The Einstein-Hilbert action, for example, is invariant under coordinate transformations which act on the metric according to

在上一节中，我们推导了单自由度标量场论的韦特里希方程 (44)。将其推广到规范场和费米子在概念上是直接的，但在引力语境下，该构造面临两个概念层面的障碍。首先，我们基于广义相对论对经典引力的理解表明，引力相互作用通过时空曲率传递。这意味着时空本身成为动力学客体，在量子理论语境下还是涨落客体。因此，提供动力学舞台的固定非动力学时空这一概念在此处不再成立。这就引出了如何定义粗粒化尺度 k 的问题。其次，引力具有规范理论的若干性质。例如，爱因斯坦-希尔伯特作用量在坐标变换下不变，坐标变换对度规的作用满足

$$\delta g_{\mu\upsilon} \equiv \mathcal{L}_{\upsilon} g_{\mu\upsilon} = \upsilon^{\rho} \partial_{\rho} g_{\mu\upsilon} + \left(\partial_{\mu} \upsilon^{\rho}\right) g_{\rho\upsilon} + \left(\partial_{\upsilon} \upsilon^{\rho}\right) g_{\rho\mu}. \tag{45}$$

Here $\mathcal{L}_{\upsilon}$ denotes the Lie derivative along the generating vector field υ^{μ} . In order to ensure that the generating functional Z_k sums over physically inequivalent configurations only, one has to introduce a suitable gauge-fixing condition. By construction, the gauge-fixing term breaks the invariance under the transformations (45). As a consequence, the effective (average) action may loose this symmetry, leading to a proliferation of interaction monomials which could be generated along the RG flow.

此处 $\mathcal{L}_{\upsilon}$ 表示沿生成矢量场 υ^{μ} 的李导数。为了确保生成泛函 Z_k 仅对物理上不等价的构型求和，必须引入合适的规范固定条件。根据构造，规范固定项会破坏变换 (45) 下的不变性。结果，有效 (平均) 作用量可能失去该对称性，导致 RG 流过程中可能产生大量相互作用单项式。

Following the seminal work by Reuter [11], both of these conceptual difficulties can be overcome by resorting to the background field method. This procedure splits the (Euclidean) quantum metric $g_{\mu\nu}$ into a generic (but non-fluctuating) background metric $\bar{g}_{\mu\nu}$ and fluctuations around this background $h_{\mu\nu}$. There is no requirement that the latter are small. The decomposition can then be implemented either through a linear or an exponential split (see [54, 55] for a detailed discussion):

遵循罗伊特开创性工作 [11]，这两个概念难题都可以借助背景场方法解决。该方法将 (欧几里得) 量子度规 $g_{\mu\nu}$ 分解为一般的 (但非涨落的) 背景度规 $\bar{g}_{\mu\nu}$ ，以及围绕该背景的涨落 $h_{\mu\nu}$ 。后者无需满足小涨落的要求。分解可以通过线性分解或指数分解实现 (详细讨论见 [54,55]):

$$g_{\mu\upsilon} = \bar{g}_{\mu\upsilon} + h_{\mu\upsilon}, \; g_{\mu\upsilon} = \bar{g}_{\mu\alpha}\left(e^{h}\right)^{\alpha}{}_{\upsilon}. \tag{46}$$

Here we follow the standard convention that indices are raised and lowered with the background metric, i.e., $h^{\mu}{}_{\nu} = \bar{g}^{\mu\alpha} h_{\alpha\nu}$, etc. While these decompositions agree to leading order in the fluctuation field, they actually define different theories, since they do not cover the same space of quantum fluctuations. Heuristically, this can be argued based on the observation that the linear split allows for $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ having different signatures, while in the exponential split this is not the case [56]. This is also confirmed by computing properties of the Reuter fixed point in $d = 2 + \varepsilon$ dimensions [57]. In the following, we will adopt the linear split for simplicity.

此处我们遵循标准约定: 指标用背景度规升降，即 $h^{\mu}{}_{\nu} = \bar{g}^{\mu\alpha}h_{\alpha\nu}$ ，依此类推。虽然这些分解在涨落场的领头阶一致，但它们实际上定义了不同的理论，因为它们覆盖的量子涨落空间并不相同。启发式地，这一点可以基于以下观察论证: 线性分解允许 $g_{\mu\nu}$ 和 $\bar{g}_{\mu\nu}$ 具有不同符号，而指数分解中不存在这种情况 [56]。这也得到了 $d = 2 + \varepsilon$ 维罗伊特定点性质计算的证实 [57]。下文我们为简便起见采用线性分解。

The background metric then allows to quantize metric fluctuations along the lines of a quantum field theory in a curved spacetime. Moreover, it allows to circumvent the conceptual difficulties discussed above as follows. Firstly, it provides the basis for separating fluctuations into "high-" and "low-" momentum modes relative to the coarse-graining scale in a purely geometric way. Taking the background to be compact and introducing the Laplacian $\Delta \equiv -\bar{g}^{\mu\upsilon}\bar{D}_{\mu}\bar{D}_{\upsilon}$ constructed from the background metric, one can obtain the ordered set of eigenmodes

背景度规使得我们可以按照弯曲时空量子场论的思路对度规涨落进行量子化。此外,它可以按如下方式绕开上述概念难题。首先, 背景度规为以纯几何方式, 相对于粗粒化尺度将涨落分离为“高”动量模和“低”动量模提供了基础。取背景为紧致流形, 引入由背景度规构造的拉普拉斯算符 $\Delta \equiv -\bar{g}^{\mu\upsilon}\bar{D}_{\mu}\bar{D}_{\upsilon}$ ，我们可以得到本征模的有序集合

$$\Delta h^{n}_{\mu\upsilon} = E_n h^{n}_{\mu\upsilon},\ n = 0, 1, \cdots, \tag{47}$$

with $E_0 \leq E_1 \leq E_2 \leq \cdots$. Fluctuations with $E_n \lesssim k^2$ are then considered "long-range" and are suppressed by the regulator, while "short-range" fluctuations characterized by $E_n \gtrsim k^2$ are integrated out without suppression factor. Practically, this is achieved by generalizing (28) to

满足 $E_0 \leq E_1 \leq E_2 \leq \cdots$。满足 $E_n \lesssim k^2$ 的涨落被视为“长程”涨落，会被调节器抑制，而以 $E_n \gtrsim k^2$ 为特征的“短程”涨落则在没有抑制因子的情况下被积掉。这在实操中通过推广 (28) 得到如下形式实现:

$$\Delta S_k\left[h; \bar{g}\right] = \frac{1}{2}\int d^d x\sqrt{\bar{g}}\left[h_{\mu\upsilon}(x)\,\mathcal{R}_k^{\mu\upsilon\alpha\beta}(\Delta)\,h_{\alpha\beta}(x)\right]. \tag{48}$$

The switch from R_k to $\mathcal{R}_k$ anticipates that, in general, the regulator is a matrix in field space carrying a nontrivial tensor structure. Note that (48) is quadratic in the fluctuation field: specifically, $\mathcal{R}_k^{\mu\upsilon\alpha\beta}(\Delta)$ is independent of the fluctuation field and depends on $\bar{g}_{\mu\nu}$ only. This property is essential in order to arrive at a FRGE of the form (44).

从 R_k 到 $\mathcal{R}_k$ 的转换预示了，一般而言调节器是场空间中的矩阵，具有非平凡张量结构。注意 (48) 是涨落场的二次型: 具体来说，$\mathcal{R}_k^{\mu\upsilon\alpha\beta}(\Delta)$ 与涨落场无关，仅依赖于 $\bar{g}_{\mu\nu}$。这个性质对于得到 (44) 形式的 FRGE 至关重要。

Secondly, the linear split allows to realize the transformation (45) in two distinct ways. Quantum gauge transformations (δ^Q) keep $\bar{g}_{\mu\nu}$ fixed and attribute the transformation of $g_{\mu\nu}$ to the fluctuation field

其次，线性分解可以通过两种不同方式实现变换 (45)。量子规范变换 (δ^Q) 保持 $\bar{g}_{\mu\nu}$ 固定，将 $g_{\mu\nu}$ 的变换归给涨落场

$$\delta^{Q}\bar{g}_{\mu\nu}=0,\ \delta^{Q}h_{\mu\nu}=\mathcal{L}_{v}\left(\bar{g}_{\mu\nu}+h_{\mu\nu}\right). \tag{49}$$

It is this transformation that must be gauge-fixed. In addition, one can define background gauge transformations $\left(\delta^{B}\right)$ where each field transforms as a tensor of the corresponding rank

正是这个变换需要做规范固定。此外，还可以定义背景规范变换 $\left(\delta^{B}\right)$ ，其中每个场都变换为对应秩的张量

$$\delta^{B}\bar{g}_{\mu v}=\mathcal{L}_{v}\bar{g}_{\mu v},\ \delta^{B}h_{\mu v}=\mathcal{L}_{v}h_{\mu v}. \tag{50}$$

This transformation can be maintained as an auxiliary symmetry by resorting to the class of background covariant gauges. Following the Faddeev-Popov procedure, the gauge fixing is implemented by supplementing the gravitational action $S[g]$ by a gauge-fixing term

通过选择背景协变规范类，该变换可以作为辅助对称性保留下来。遵循法捷耶夫-波波夫步骤，规范固定通过给引力作用量 $S[g]$ 补充规范固定项实现

$$S^{\mathrm{gf}}[h;\bar{g}]=\frac{1}{2\alpha}\int d^{d}x\sqrt{\bar{g}}\bar{g}^{\mu v}F_{\mu}F_{\mu}. \tag{51}$$

Here, α is a free parameter and the gauge-fixing condition $F_{\mu}[h;\bar{g}]$ transforms as a rank-one tensor with respect to (50).

此处，α 是自由参数，规范固定条件 $F_{\mu}[h;\bar{g}]$ 相对于 (50) 按一阶张量变换。

The gauge-fixing term is accompanied by the action for the Faddeev-Popov ghost and anti-ghost fields C^{μ} and $\bar{C}_{\mu}$

规范固定项附带了法捷耶夫-波波夫鬼场和反鬼场的作用量 C^{μ} 和 $\bar{C}_{\mu}$

$$S^{\text{ ghost }}\left[h,\bar{C},C;\bar{g}\right]=-\sqrt{2}\int d^{d}x\sqrt{\bar{g}}\bar{C}_{\mu}\bar{g}^{\mu v}\frac{\delta F_{v}}{\delta h_{\alpha\beta}}\mathcal{L}_{C}\left(\bar{g}_{\alpha\beta}+h_{\alpha\beta}\right). \tag{52}$$

This action exponentiates the Faddeev-Popov determinant

该作用量对应法捷耶夫-波波夫行列式的指数形式

$$\det\mathcal{M}=\det\left[\frac{\delta F_{\mu}}{\delta v^{v}}\right]=\int\mathcal{D}C^{\mu}\mathcal{D}\bar{C}_{v}e^{-\int\bar{C}\mathcal{M}C}. \tag{53}$$

At this point we have all the ingredients to write down the analogue of the generating functional (33) in the context of gravity

至此，我们已经具备了所有要素，可以写出引力语境下生成泛函 (33) 的对应形式

$$\exp\left(W_{k}[J;\bar{g}]\right)=\frac{1}{N}\int\mathcal{D}h_{\alpha\beta}\mathcal{D}C^{\mu}\mathcal{D}\bar{C}_{v}\exp\left\{-S[\bar{g}+h]-S^{\mathrm{gf}}[h;\bar{g}]\right.$$

$$-S^{\text{ghost}}\left[h,\bar{C},C;\bar{g}\right]-\Delta S_k\left[h,\bar{C},C;\bar{g}\right]+S^{\text{source}}\Big\}.$$

(54)

Here $S[g]$ denotes a generic action built from the metric $g_{\mu\nu}$, invariant under (45), $S^{\text{gf}}[h;\bar{g}]$ and $S^{\text{ghost}}\left[h,\bar{C},C;\bar{g}\right]$ are the gauge-fixing and ghost actions given in Eqs. (51) and (52), and $\Delta S_k\left[h,\bar{C},C;\bar{g}\right]$ is the IR regulator (48) extended by a k -dependent mass term for the ghost fields. Finally,

此处 $S[g]$ 表示由度规 $g_{\mu\nu}$ 构造的一般作用量，在 (45) 下不变，$S^{\text{gf}}[h;\bar{g}]$ 和 $S^{\text{ghost}}\left[h,\bar{C},C;\bar{g}\right]$ 分别是式 (51) 和 (52) 给出的规范固定作用量和鬼作用量，$\Delta S_k\left[h,\bar{C},C;\bar{g}\right]$ 是 (48) 的红外调节器，它扩展了一个依赖 k 的鬼场质量项。最后，

$$S^{\text{source}}=\int d^dx\sqrt{\bar{g}}\left\{t^{\mu\nu}h_{\mu\nu}+\bar{\sigma}_\mu C^\mu+\sigma^\mu\bar{C}_\mu\right\} \tag{55}$$

introduces sources for the quantum field, which we collectively label by $J\equiv\left(t^{\mu\nu},\sigma^\mu,\bar{\sigma}_\mu\right)$.

引入了量子场的源，我们将其统一记为 $J\equiv\left(t^{\mu\nu},\sigma^\mu,\bar{\sigma}_\mu\right)$。

The construction of the effective average action then proceeds analogously to the scalar case. Taking functional derivatives of $W_k[J;\bar{g}]$ with respect to the sources gives the expectation values of the fluctuation fields

有效平均作用量的构造随后可以类比标量情形完成。对源对 $W_k[J;\bar{g}]$ 取泛函导数，即可得到涨落场的期望值

$$\langle h_{\mu\nu}\rangle=\frac{1}{\sqrt{\bar{g}}}\frac{\delta W_k}{\delta t^{\mu\nu}},\ \langle\bar{C}_\mu\rangle=\frac{1}{\sqrt{\bar{g}}}\frac{\delta W_k}{\delta\sigma^\mu},\ \langle C^\mu\rangle=\frac{1}{\sqrt{\bar{g}}}\frac{\delta W_k}{\delta\bar{\sigma}_\mu}. \tag{56}$$

In a slight abuse of notation we then use the same labels for the mean and quantum fields, identifying

此处我们稍作不规范的记号混用，对平均场和量子场使用相同标记，将其等同为

$$h_{\mu\nu}=\langle h_{\mu\nu}\rangle,\ C^\mu=\langle C^\mu\rangle,\ \bar{C}_\mu=\langle\bar{C}_\mu\rangle,\ g_{\mu\nu}=\langle\bar{g}_{\mu\nu}+h_{\mu\nu}\rangle. \tag{57}$$

We then assume again that the field-source relations (56) can be solved for the sources as functions of the mean field. The effective average action is then again defined as the modified Legendre transform of W_k :

我们再次假设场-源关系 (56) 可以解出作为平均场函数的源。有效平均作用量由此再次定义为 W_k 的修正勒让德变换:

$$\Gamma_k[\Phi;\bar{g}]=\int d^dx\sqrt{\bar{g}}\left\{t^{\mu\nu}h_{\mu\nu}+\bar{\sigma}_\mu C^\mu+\sigma^\mu\bar{C}_\mu\right\}-W_k[J;\bar{g}]-\Delta S_k[\Phi;\bar{g}]. \tag{58}$$

Here we used $\Phi=\left(h,\bar{C}_\mu,C^\mu\right)$ to denote the collection of expectation values.

此处我们用 $\Phi = \left(h, \bar{C}_\mu, C^\mu\right)$ 表示所有期望值的集合。

The key property of the effective average action (58) is that its dependence on the coarse-graining scale k is again governed by a formally exact functional renormalization equation taking the form (44). Its derivation essentially follows the one for the scalar theory. Taking the derivative of (58) with respect to the RG time t and expressing the right-hand side in terms of the Hessian of $\Gamma_k\left[\Phi;\bar{g}\right]$, one finds [11]

有效平均作用量 (58) 的核心性质是，它对粗粒化标度 k 的依赖同样由一个形式精确、形如 (44) 的泛函重整化方程支配。其推导基本遵循标量理论的推导过程。对 (58) 关于 RG 时间 t 求导，再将右侧用 $\Gamma_k\left[\Phi;\bar{g}\right]$ 的黑塞矩阵表示，可得 [11]

$$\partial_t \Gamma_k\left[\Phi;\bar{g}\right] = \frac{1}{2}\,\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}_{hh}\left(\partial_t \mathcal{R}_k\right)_{hh}\right]$$

$$-\frac{1}{2}\,\mathrm{Tr}\left[\left\{\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}_{\bar{C}C} - \left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}_{C\bar{C}}\right\}\left(\partial_t \mathcal{R}_k\right)_{\bar{C}C}\right].$$

(59)

Here the matrix elements constituting the Hessian of Γ_k are defined via

此处构成 Γ_k 黑塞矩阵的矩阵元定义为

$$\left(\Gamma_k^{(2)}\right)_{ij}(x,y) \equiv \frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\frac{\delta^2 \Gamma_k}{\delta \Phi^i(x)\,\delta \Phi^j(y)}. \tag{60}$$

For the Grassmann-valued (anti-commuting) fields in the ghost sector, we adopt the convention that matrix elements are defined in terms of left derivatives, i.e.,

对于鬼区的格拉斯曼值 (反对易) 场，我们约定矩阵元由左导数定义，即

$$\left(\left(\Gamma_k^{(2)}\right)_{\bar{C}C}\right)_\mu{}^v(x,y) = \frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta}{\delta C^\mu(x)}\frac{1}{\sqrt{\bar{g}(y)}}\frac{\delta}{\delta \bar{C}_v(y)}\Gamma_k\left[\Phi;\bar{g}\right]. \tag{61}$$

Introducing a supertrace STr which includes a sum over all fluctuation fields as well as a minus sign for Grassmann-valued degrees of freedom, Eq. (59) can again be written in compact form,

引入超迹 STr，它涵盖对所有涨落场的求和，且对格拉斯曼取值的自由度带一个负号后，式 (59) 可再次改写为紧致形式:

$$\partial_t \Gamma_k\left[\Phi;\bar{g}\right] = \frac{1}{2}\,\mathrm{STr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}\partial_t \mathcal{R}_k\right]. \tag{62}$$

This equation maintains all the properties discussed in the context of the scalar theory. It is the central result of this section and constitutes the starting point for investigating the Wilsonian renormalization group flow of gravity. Notably, its use is not limited to the case where the gravitational degrees of freedom are encoded in metric fluctuations. It is also applicable to formulations building on different sets of degrees of freedom, including unimodular gravity, the Hilbert-Palatini formulation, the Arnowitt-Deser-Misner (ADM) decomposition of the metric degrees of freedom, and also Hořava-Lifshitz gravity. This makes (62) a powerful and rather universal tool to study the quantum properties of gravity beyond perturbation theory and its use in practical computations will be discussed in section ”The Einstein-Hilbert Truncation.”

该方程保留了标量理论讨论中所有性质。它是本节的核心结果，也是研究引力威尔逊重整化群流的出发点。值得注意的是，它的适用范围并不局限于引力自由度由度规涨落描述的情形，它也适用于基于不同自由度集合的表述，包括 unimodular 引力、希尔伯特-帕拉蒂尼表述、阿尔诺维特-德瑟-米斯纳 (ADM) 度规自由度分解，以及霍拉瓦-利夫希茨引力。这使得式 (62) 成为研究非微扰引力量子性质的强大且相当普适的工具，其实际计算中的应用将在“爱因斯坦-希尔伯特截断”一节讨论。

At this point the following conceptual clarifications are in order. At first sight the introduction of a background metric seems to contradict the idea of background independence intrinsic to general relativity. This is not the case though. Keeping $\bar{g}_{\mu\nu}$ generic essentially corresponds to quantizing the theory in all backgrounds simultaneously. Subsequently, one can then evoke a dynamical principle determining $\bar{g}_{\mu\nu}$. In this way one retains background independence even in the presence of a background metric. This viewpoint underlies the concept of self-consistent backgrounds developed in [58, 59] .

在此需要做一些概念澄清。乍看之下，引入背景度规似乎与广义相对论内禀的背景无关性思想矛盾，但事实并非如此。保持 $\bar{g}_{\mu\nu}$ generic 本质上对应同时对所有背景下的理论做量子化，之后我们可以引入动力学原理来确定 $\bar{g}_{\mu\nu}$ 。通过这种方式，即使引入了背景度测，我们依然保留了背景无关性。这一观点是 [58, 59] 中发展的自洽背景概念的基础。

Common Approximation Schemes

常用近似方案

The Wetterich equation (62) constitutes a formally exact equation. Finding exact solutions to it is equivalent to carrying out the functional integral (1). This is extremely ambitious though and usually cannot be carried out exactly. Thus, one has to resort to approximations.

维特里希方程 (62) 是一个形式上的精确方程。找到它的精确解等价于完成泛函积分 (1)。但这要求极高，通常无法精确求解，因此必须借助近似方法。

Probably, the most prominent approximation is perturbation theory. In this case the standard result is recovered by neglecting the k -dependence of $\Gamma_k^{(2)}$ on the righthand side of Eq. (62) and approximating $\Gamma_k^{(2)} \to S_\Lambda^{(2)}$ with S_Λ the bare action defined at the UV scale Λ . This approximation turns the trace into a total derivative

最著名的近似方法当属微扰论。该方案通过忽略式 (62) 右侧 $\Gamma_k^{(2)}$ 对 k 的依赖，并用紫外标度 Λ 处定义的裸作用量 S_Λ 近似 $\Gamma_k^{(2)} \to S_\Lambda^{(2)}$ ，最终得到标准结果。该近似将迹转化为全导数

$$\partial_t \Gamma_k \simeq \frac{1}{2} \partial_t \operatorname{Tr}\left[\ln\left(S_\Lambda^{(2)} + \mathcal{R}_k\right)\right]. \tag{63}$$

Here and in the following we use $\simeq$ to indicate an approximation of the exact flow. Integrating this equation from the UV scale down to $k = 0$ and assuming that the regulator vanishes at the boundaries then yields the standard formula for the one-loop effective action

下文中我们统一用 $\simeq$ 表示精确流的近似。将该方程从紫外标度积分到 $k = 0$，并假设调节器在边界处为零，即可得到单圈有效作用量的标准公式

$$\Gamma^{1-\mathrm{loop}} = S_\Lambda + \frac{1}{2} \mathrm{Tr} \left[\ln S_\Lambda^{(2)} \right]. \tag{64}$$

The investigation of RG fixed points typically builds on non-perturbative approximation schemes though. The basic idea is to start from the exact flow and project it onto a subspace spanned by a finite (or even infinite) set of interaction monomials $\mathcal{O}_i$. In the setup introduced in section "The Asymptotic Safety Mechanism," this amounts to truncating the sum in Eq. (4) to a finite set

不过重整化群不动点的研究通常建立在非微扰近似方案之上。其基本思路是从精确流出发，将其投影到由有限 (甚至无穷) 组相互作用单项式 $\mathcal{O}_i$ 张成的子空间上。在“渐近安全机制”一节介绍的框架中，这相当于将式 (4) 中的求和截断为有限项

$$\Gamma_k \simeq \sum_{i=1}^{N} \bar{u}_i (k) \mathcal{O}_i \tag{65}$$

These types of approximations can be set up systematically, either in the form of a derivative expansion or a vertex expansion. These commonly used non-perturbative approximation schemes will be discussed in sections "Derivative and Curvature Expansion" and "Incorporating Higher-Order Interaction Vertices," respectively.

这类近似可以系统构造，分为导数展开和顶点展开两种形式。我们将分别在“导数与曲率展开”和“包含高阶相互作用顶点”两节中讨论这些常用的非微扰近似方案。

Derivative and Curvature Expansion

导数与曲率展开

When developing non-perturbative approximation schemes, it is important to appreciate that Γ_k depends on two metric arguments $g_{\mu\upsilon}$ and $\bar{g}_{\mu\upsilon}$. The dependence on $g_{\mu\nu}$ can be traded for the fluctuations $h_{\mu\nu}$ by substituting the linear split (46). Structurally, it is then convenient to organize the contributions in Γ_k according to their transformation properties with respect to the background and quantum gauge transformations

在建立非微扰近似方案时，必须注意 Γ_k 依赖于两个度规参数 $g_{\mu\upsilon}$ 和 $\bar{g}_{\mu\upsilon}$。通过代入线性拆分式 (46)，对 $g_{\mu\nu}$ 的依赖可以转化为涨落 $h_{\mu\nu}$ 的形式。从结构上看，根据各贡献关于背景规范变换和量子规范变换的变换性质对 Γ_k 中的项进行整理会十分方便

$$\Gamma_k \left[g, \bar{g}, \bar{C}, C \right] = \overline{\Gamma}_k \left[g \right] + \hat{\Gamma}_k \left[g, \bar{g} \right] + \Gamma_k^{\mathrm{gf}} \left[g, \bar{g} \right] + \Gamma_k^{\mathrm{ghost}} \left[g, \bar{g}, \bar{C}, C \right]. \tag{66}$$

Here $\Gamma_k^{\mathrm{gf}} \left[g, \bar{g} \right]$ and $\Gamma_k^{\mathrm{ghost}} \left[g, \bar{g}, C, \bar{C} \right]$ are the standard gauge-fixing and ghost terms. The subscript k thereby indicates that these sectors can contain k -dependent couplings, as, e.g., a wave-function renormalization for the ghost fields. The contribution $\overline{\Gamma}_k[g]$ collects all terms constructed from $g_{\mu\nu}$ only. By construction, $\overline{\Gamma}_k[g]$ is then invariant with respect to both background and quantum gauge transformations

此处 $\Gamma_k^{\mathrm{gf}}[g,\bar{g}]$ 和 $\Gamma_k^{\mathrm{ghost}}\left[g,\bar{g},C,\bar{C}\right]$ 是标准的规范固定项和鬼项。下标 k 表明这些部分可以包含依赖 k 的耦合，例如鬼场的波函数重整化。贡献项 $\overline{\Gamma}_k[g]$ 汇集了所有仅由 $g_{\mu\nu}$ 构造的项，根据构造，$\overline{\Gamma}_k[g]$ 同时对背景规范变换和量子规范变换保持不变

$$\delta^B\overline{\Gamma}_k[g]=0,\ \delta^Q\overline{\Gamma}_k[g]=0. \tag{67}$$

The terms contained in $\hat{\Gamma}_k[g,\bar{g}]$ genuinely depend on both arguments. It collects the "off-diagonal" contributions and satisfies

$\hat{\Gamma}_k[g,\bar{g}]$ 包含的项确实同时依赖于两个参数。它汇集了所有“非对角”贡献，且满足

$$\hat{\Gamma}_k[g,g]=0. \tag{68}$$

A rather broad class of approximations based on (66) truncates the effective average action by setting $\hat{\Gamma}_k[g,\bar{g}] \simeq 0$. Commonly, these approximations are referred to as single-metric approximations [60-62]. Most approximations along these lines also work with a classical ghost sector, setting $\Gamma_k^{\mathrm{ghost}}\left[g,\bar{g},\bar{C},C\right] \simeq S^{\mathrm{ghost}}\left[g,\bar{g},\bar{C},C\right]$.

基于式 (66) 的一大类近似通过令 $\hat{\Gamma}_k[g,\bar{g}]\simeq 0$ 截断有效平均作用量，这类近似通常被称为单度量近似 [60-62]。大多数这类近似也采用经典鬼区，令 $\Gamma_k^{\mathrm{ghost}}\left[g,\bar{g},\bar{C},C\right]\simeq S^{\mathrm{ghost}}\left[g,\bar{g},\bar{C},C\right]$。

Building on the results by Fulling, King, Wybourne, and Cummins [63] (further elaborated on in [64]), one can systematically construct a basis $\mathcal{O}_i[g]$ in which $\overline{\Gamma}_k[g]$ can be expanded. The explicit construction of the independent basis elements needs to take into account redundancies due to the Bianchi identity $D_{[\mu}R_{\alpha\beta]\gamma\delta}=0$. In addition, low-dimensional cases are subject to additional simplifications, e.g., due to the vanishing of the Weyl tensor in $d=3$.

基于 Fulling、King、Wybourne 和 Cummins 的研究结果 [63](文献 [64] 对其做了进一步拓展), 我们可以系统地构造一个能够展开 $\overline{\Gamma}_k[g]$ 的基 $\mathcal{O}_i[g]$ 。构造独立基元时需要考虑比安基恒等式 $D_{[\mu}R_{\alpha\beta]\gamma\delta}=0$ 带来的冗余性。此外，低维情形存在额外的简化，例如 $d=3$ 中外尔张量为零。

The symmetries of $\overline{\Gamma}_k[g]$ dictate that the corresponding monomials are built from the Riemann tensor $R_{\mu\upsilon\rho\sigma}$, its contractions, and covariant derivatives D_μ acting on the curvature tensors. Convenient building blocks for the basis elements are then provided either by the Riemann basis

$\overline{\Gamma}_k[g]$ 的对称性决定了对应的单项式由黎曼张量 $R_{\mu\upsilon\rho\sigma}$ 、它的缩并以及作用在曲率张量上的协变导数 D_μ 构造而成。因此，基元可以方便地由黎曼基提供构造单元

$$\mathcal{O}_i[g]=\mathcal{O}_i\left[\sqrt{g},R,R_{\mu\nu},R_{\mu\nu\rho\sigma},D_\mu\right] \tag{69}$$

or the Weyl basis

或者外尔基

$$\mathcal{O}_i[g] = \mathcal{O}_i\left[\sqrt{g}, R, R_{\mu\nu}, C_{\mu\nu\rho\sigma}, D_\mu\right]. \tag{70}$$

The two choices are related by the identity

两种选择通过下述恒等式关联

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{d-2}\left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) + \frac{2}{(d-1)(d-2)}Rg_{\mu[\rho}g_{\sigma]\nu}.$$

(71)

In terms of structural aspects, it is often useful to work in the Weyl basis, since this choice disentangles the contributions of the higher-derivative terms to the flat-space graviton propagator.

从结构角度来看，采用外尔基通常更有用，因为这种选择可以解耦高导数项对平空间引力子传播子的贡献。

The expansion of $\bar{\Gamma}_k[g]$ can be organized systematically by counting the number of spacetime derivatives n contained in the monomial $\mathcal{O}_i[g] \equiv \int d^dx\sqrt{g}\tilde{\mathcal{O}}_l^n[g]$. The index set $\{i\} \mapsto \{n, l\}$ where l enumerates the basis elements occurring at a fixed order n; see Table 1 for examples. This scheme is called the derivative expansion of $\bar{\Gamma}_k[g]$. The basis elements appearing at the lowest orders are given in Table 1. The number of independent basis elements increases significantly with each order in the derivative expansion. This expansion scheme provides a good ordering principle when studying the ”low-energy” properties of the theory. For fixed points which are Gaussian or ”almost-Gaussian,” the power counting also provides a good guiding principle whether a given operator is relevant or irrelevant.

$\bar{\Gamma}_k[g]$ 的展开可以通过统计单项式 $\mathcal{O}_i[g] \equiv \int d^dx\sqrt{g}\tilde{\mathcal{O}}_l^n[g]$ 中包含的时空导数 n 的数量进行系统整理。指标集 $\{i\} \mapsto \{n, l\}$ 中，l 列举出固定阶数 n 下出现的基元；示例见表 1。该方案被称为 $\bar{\Gamma}_k[g]$ 的导数展开。低阶出现的基元已列于表 1。在导数展开中，独立基元的数量会随着每一阶显著增加。研究理论的“低能”性质时，该展开方案提供了良好的排序原则。对于高斯或“近似高斯”不动点，幂计数也能为判断给定算符是相关还是不相关提供良好的指导原则。

A conceptual shortcoming of the derivative expansion is that truncating the series of terms contributing to the gravitational propagator induces potentially spurious poles [65]. The reason is that the approximation intrinsic to the derivative expansion leads to inverse propagators which are polynomial in the momentum. Hence, it is difficult to address questions about stability and the potential presence of ghosts within this approximation [66,67].

导数展开的一个概念缺陷是，截断对引力传播子有贡献的项序列会诱发潜在的伪极点 [65]。原因在于，导数展开本身固有的近似会导致逆传播子是动量的多项式。因此，在该近似框架内很难研究稳定性问题以及鬼粒子可能存在的问题 [66,67]。

As stressed in [68], this feature can be bypassed by switching to a curvature expansion. The basic idea is to collect the covariant derivatives appearing in interaction monomials in operator-valued functions, called form factors. These capture the dependence of propagators and interaction vertices on the (generalized) momenta of the fields and can also be defined in an arbitrary curved background spacetime. Building on the examples given in Table 1, the form factors appearing at the lowest nontrivial order in the curvature expansion arise from combining the terms in the columns with $l = 1$ and $l = 2$:

正如文献 [68] 所强调的，切换到曲率展开就可以避开这一问题。其核心思路是将相互作用单项式中出现的协变导数归集为算符值函数，即形状因子。形状因子可以描述传播子和相互作用顶点对场的 (广义) 动量的依赖关系，还可以在任意弯曲背景时空中定义。基于表 1 给出的例子，曲率展开最低非平凡阶出现的形状因子来源于将对应列中含 $l = 1$ 和 $l = 2$ 的项组合得到:

$$\sum_{i=0} \bar{u}^n (k) R\Delta^n R \mapsto RW_k^R (\Delta) R$$

$$\sum_{i=0} \bar{u}^n (k) C_{\mu\upsilon\rho\sigma}\Delta^n C^{\mu\upsilon\rho\sigma} \mapsto C_{\mu\upsilon\rho\sigma}W_k^C (\Delta) C^{\mu\upsilon\rho\sigma}. \tag{72}$$

Notably, there are only two form factors appearing at second order in the spacetime curvature. A potential third function $R_{\mu\upsilon}W_k^{\mathrm{Ric}} (\Delta) R^{\mu\upsilon}$ can be mapped to (72) and higher-order curvature terms by applying the Bianchi identity. The functions $W_{k=0}^C (\Delta)$ and $W_{k=0}^R (\Delta)$ fix the graviton propagator in a flat background. From Table 1, it is also apparent that there is no form factor at first order in the curvature expansion. Any derivatives acting on R would lead to a surface term. As a consequence, Newton’s constant G_0 (and also the cosmological constant Λ_0) cannot carry a dependence on the physical momenta of the field.

值得注意的是，时空曲率二阶仅出现两个形状因子。利用比安基恒等式，可以将潜在的第三个函数 $R_{\mu\upsilon}W_k^{\mathrm{Ric}} (\Delta) R^{\mu\upsilon}$ 映射到 (72) 和更高阶曲率项。函数 $W_{k=0}^C (\Delta)$ 和 $W_{k=0}^R (\Delta)$ 确定了平坦背景下的引力子传播子。从表 1 还可以明显看出，曲率展开的一阶不存在形状因子。任何作用在 R 上的导数都会给出一个表面项。因此，牛顿常数 G_0 (以及宇宙学常数 Λ_0) 不依赖于场的物理动量。

The k -dependence of a form factor can again be obtained by substituting the corresponding ansatz for Γ_k into the Wetterich equation and projecting the flow on the corresponding subspace. In general, this results in a nonlinear integrodifferential equation for the unknown functions; see Table 2. Solving these equations either numerically or by employing pseudospectral methods then allows to obtain information on the graviton propagator and momentum dependence of interaction vertices; see [69] for pioneering work in this direction.

形状因子对 k 的依赖可以通过以下方式得到: 将 Γ_k 对应的试探解代入韦特里希方程，再把流投影到对应子空间。一般而言，这会得到未知函数的一个非线性积分微分方程，参见表 2。通过数值方法或伪谱方法求解这些方程，就可以得到引力子传播子和相互作用顶点动量依赖的相关信息，该方向的开创性工作参见文献 [69]。

Incorporating Higher-Order Interaction Vertices

纳入高阶相互作用顶点

The background approximation evaluates the Wetterich equation at zeroth order in the fluctuation field. This class of approximations can then be extended systematically by taking into account higher orders of the fluctuation field. This is the idea behind the bimetric computations initiated in [60-62] and the fluctuation approach reviewed in [34]. It can be implemented systematically by performing a vertex expansion of $\Gamma_k\,[h;\bar{g}]$ in powers of the fluctuation field.

背景近似在涨落场的零阶计算 Wetterich 方程。这类近似可以通过考虑涨落场的更高阶来系统地推广。这就是 [60-62] 中开创的双度量计算以及 [34] 中综述的涨落方法的核心思想。它可以通过将 $\Gamma_k\,[h;\bar{g}]$ 按涨落场的幂次做顶点展开来系统实现。

$$\Gamma_k\,[h;\bar{g}] = \sum_{n,l} \frac{1}{n!} \int d^d x \Gamma_k^{l;\mu_1 \upsilon_1 \cdots \mu_n \upsilon_n}\,[\bar{g}]\, h_{\mu_1 \upsilon_1} \cdots h_{\mu_n \upsilon_n}. \tag{73}$$

The discussion of the ghost contributions follows the same lines, but is suppressed for the sake of readability. Here l enumerates the set of independent tensor structures contracting n powers of the fluctuation fields. Note that all dependence on the background metric is stored in $\Gamma_k^{l;\mu_1 \upsilon_1 \cdots \mu_n \upsilon_n}\,[\bar{g}]$. Similarly to (69) and (70), the vertices can be built from $\sqrt{\bar{g}}$ and background curvature tensors and their contractions, as well as the background covariant derivative. By construction $\Gamma_k^{l;\mu_1 \upsilon_1 \cdots \mu_n \upsilon_n}\,[\bar{g}]$ transforms as a tensor of the corresponding rank with respect to background gauge transformations. Since the expansion captures contributions from both $\overline{\Gamma}_k\,[g]$ and $\hat{\Gamma}_k\,[g,\bar{g}]$, quantum gauge invariance is broken and the classification of admissible vertices is significantly more complicated than in the single-metric case. Prototypical examples of terms appearing in the vertex expansion can be obtained from expanding the gauge-fixed Einstein-Hilbert action in powers of $h_{\mu\upsilon}$. Explicit examples can then be found in Eqs. (95) and (96).

鬼贡献的讨论遵循相同的思路，但为了可读性在此省略。此处 l 枚举出了收缩 n 次幂涨落场的独立张量结构集合。注意所有对背景度规的依赖都存储在 $\Gamma_k^{l;\mu_1 \upsilon_1 \cdots \mu_n \upsilon_n}\,[\bar{g}]$ 中。与式 (69) 和 (70) 类似，顶点可以由 $\sqrt{\bar{g}}$、背景曲率张量及其缩并，还有背景协变导数构造得到。根据构造，$\Gamma_k^{l;\mu_1 \upsilon_1 \cdots \mu_n \upsilon_n}\,[\bar{g}]$ 相对于背景规范变换按对应秩的张量变换。由于该展开同时包含了来自 $\overline{\Gamma}_k\,[g]$ 和 $\hat{\Gamma}_k\,[g,\bar{g}]$ 的贡献，量子规范不变性被破坏，可允许顶点的分类比单度量情形复杂得多。顶点展开中出现的项的典型原型可以通过将规范固定的爱因斯坦-希尔伯特作用量按 $h_{\mu\upsilon}$ 的幂次展开得到。具体例子可以在式 (95) 和 (96) 中找到。

Table 2 Summary of the mathematical structures capturing the flow of Γ_k in different classes of approximations. Depending on the scale-dependent terms retained in Γ_k , the projected flow equations are nonlinear ordinary differential equations (ODEs), partial differential equations (PDEs), or (partial) integro differential equations (IDEs). Since fixed functionals are k-stationary solutions, their structure is encoded in differential equations which contain one variable less than the corresponding flow equation

表 2 概括了不同近似类别下描述 Γ_k 流的数学结构。根据 Γ_k 中保留的依赖标度的项，投影后的流方程可以是非线性常微分方程 (ODE)、偏微分方程 (PDE)，或是 (偏) 积分微分方程 (IDE)。由于固定泛函是 k 平稳解，其结构编码在比对应流方程少一个变量的微分方程中

Approximation of Γ_k	Structure of RG flow	Fixed points
Finite number of $\mathcal{O}_i$	ODEs	Algebraic
Field-dependent functions $f_k\,(R_1,\cdots,R_n)$	PDEs $(n+1)$ variables	PDEs n variables
Momentum-dependent form factors $f_k\,(p_1,\cdots,p_n)$	IDEs $(n+1)$ variables	IDEs n variables

The k -dependence of the vertices appearing in (73) can again be obtained from the Wetterich equation. Taking functional derivatives of (2) with respect to the fluctuation fields gives a hierarchy of equations determined by the schematic form

式 (73) 中出现的顶点对 k 的依赖可以再次从 Wetterich 方程得到。对式 (2) 关于涨落场取泛函导数，就得到了由该概略形式确定的方程组层级

$$\partial_t \Gamma_k^{(n)}[\bar{g}] = \text{Flow}\left[\Gamma_k^{(2)}[\bar{g}], \cdots, \Gamma_k^{(n+2)}[\bar{g}]\right]. \tag{74}$$

Here the superscript indicates the n th functional derivative of Γ_k with respect to the fluctuation fields; cf. (60). Background computations evaluate this hierarchy at zeroth order in n . Note that the right-hand side also depends on the higher-order vertices $\Gamma_k^{(n+1)}[\bar{g}]$ and $\Gamma_k^{(n+2)}[\bar{g}]$. The truncation of the system to a finite set of tensor structures then requires an assumption on these higher-order vertices in order to close the system. A typical strategy is to approximate the couplings appearing at the orders $(n+1)$ and $(n+2)$ by the ones appearing at the lower orders in the hierarchy.

此处上标表示对涨落场的 n 次泛函导数，其中求导对象是 Γ_k ；参见式 (60)。背景计算在 n 的零阶对该层级求值。注意右侧也依赖于高阶顶点 $\Gamma_k^{(n+1)}[\bar{g}]$ 和 $\Gamma_k^{(n+2)}[\bar{g}]$ 。将系统截断为有限个张量结构需要对这些高阶顶点做出假设才能封闭系统。典型策略是用层级中更低阶出现的耦合来近似 $(n+1)$ 和 $(n+2)$ 阶出现的耦合。

In practice, computations maintaining information about the fluctuation fields have mainly been carried out in a flat background, setting $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$. This choice gives access to powerful momentum space techniques and the hierarchy (74) can then be evaluated by employing standard Feynman diagram techniques. In particular, Eq. (73) simplifies to

实际上，保留涨落场信息的计算主要是在平直背景下进行的，此时设定 $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$ 。该选择可以利用强大的动量空间技术，并且可以通过标准费曼图技术计算层级 (74)。具体来说，式 (73) 可以简化为

$$\Gamma_k[h;\delta] = \sum_{n,l} \frac{1}{n!} \left(\prod_n \int \frac{d^d p}{(2\pi)^d} \right) \Gamma_k^{l;\mu_1\upsilon_1\cdots\mu_n\upsilon_n}(p_1,\cdots,p_n)\, h_{\mu_1\upsilon_1}(p_1)\cdots h_{\mu_n\upsilon_n}(p_n), \tag{75}$$

where the p_i are the momenta of the fluctuation fields. This has led to significant insights on the momentum dependence of the graviton two-point function [70, 71] and the momentum dependence of three- and four-point vertices [72, 73] .

其中 p_i 是涨落场的动量。该方法已为引力子两点函数 [70, 71] 的动量依赖关系，以及三点、四点顶点 [72, 73] 的动量依赖关系带来了重要认知。

Further Developments

进一步发展

The discussion of the Wetterich equation and its properties mainly followed the initial constructions [11, 20, 46] . We complete our exposition by briefly introducing two recent developments, the minimal essential scheme [74] (Section ”The Minimal Essential Scheme”) and the N -type cutoffs [75,76] (Section ”Flows in Terms of N -Type Cutoffs”).

对 Wetterich 方程及其性质的讨论主要遵循了最初的构建工作 [11, 20, 46] 。本文通过简要介绍两项最新进展来完成阐述: 最小本质方案 [74](章节“最小本质方案”) 与 N 型截断 [75,76](章节“基于 N 型截断的流”)。

The Minimal Essential Scheme

最小本质方案

Ultimately, the goal of the gravitational asymptotic safety program is the construction of observables. From this perspective, it turns out that the theory space introduced in section ”The Asymptotic Safety Mechanism,” spanned by all possible interaction monomials $\mathcal{O}_i$, contains redundancies in the sense that not all couplings appearing in this basis will also enter into the observables. A prototypical example is the wave-function renormalization of a field, which drops out from the construction of scattering amplitudes. On this basis one distinguishes between essential couplings which enter into the expressions for physical observables and inessential couplings whose values can be changed without affecting the predictions of the theory.

引力渐近安全方案的最终目标是构造可观测量。从这个角度看，“渐近安全机制”小节中引入的、由所有可能相互作用单项式 $\mathcal{O}_i$ 张成的理论空间存在冗余: 并非该基矢中出现的所有耦合都会进入可观测量。一个典型例子是场的波函数重整化，它在散射振幅的构造中会被消去。在此基础上，人们将耦合分为进入物理可观测量表达式的本质耦合，以及取值改变不影响理论预言的非本质耦合。

Typically, a change in an inessential coupling can be absorbed into a reparam-eterization of the dynamical variables. Considering an infinitesimal change in the field, $\chi \mapsto \chi + \xi[\chi]$, the underlying action transforms as

通常，非本质耦合的改变可以被动力学变量的重新参数化吸收。考虑场的无穷小变换 $\chi \mapsto \chi + \xi[\chi]$ ，基础作用量会变换为

$$S[\chi] \mapsto S[\chi] + \xi[\chi] \cdot \frac{\delta}{\delta\chi} S[\chi]. \tag{76}$$

We use the ”.” to indicate an integral over spacetime and potentially a sum over internal indices labeling the fields. This underlies the general statement that operators which are proportional to the equations of motion can be removed by a field redefinition and are thus linked with inessential couplings [77]. Generically, one can also consider finite frame transformations to a new field parameterization,

我们用“.”表示对时空的积分，也可能包含标记场的内指标求和。这支撑了一个普遍结论: 与运动方程成正比的算符可以通过场重新定义移除，因此与非本质耦合相关联 [77]。一般来说，我们也可以考虑指向新场参数化的有限标架变换,

$$\phi(x) = \phi[\chi](x), \tag{77}$$

requiring that the map is quasi-local and invertible.

要求该变换是拟局部且可逆的。

Implementing the procedure of removing inessential couplings at the level of the functional renormalization group is slightly more complicated. Since the corresponding couplings depend on the coarse-graining scale k , the field redefinitions required in this process inherit this scale dependence. Thus, the frame transformation (77) is promoted to be k -dependent

在泛函重整化群层面实现移除非本质耦合的过程要稍微复杂一些。由于对应耦合依赖于粗粒化尺度 k ，该过程所需的场重新定义也继承了这种尺度依赖性。因此，标架变换 (77) 被推广为依赖 k 的形式

$$\phi_k(x) = \phi_k[\chi](x)\,. \tag{78}$$

This effect can be accommodated by formulating the Wetterich equation in a frame-covariant way [31]

这一效应可以通过以标架协变的形式表述 Wetterich 方程来处理 [31]

$$\left(\partial_t + \Psi_k[\phi]\frac{\delta}{\delta\phi}\right)\Gamma_k[\phi] = \frac{1}{2}\,\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}\left(\partial_t + 2\Psi_k[\phi]\frac{\delta}{\delta\phi}\right)\mathcal{R}_k\right].$$

(79)

The renormalization group kernel

重整化群核

$$\Psi_k[\phi] \equiv \partial_t\phi_k[\chi] \tag{80}$$

thereby accounts for the k -dependence of the frame transformation.

由此就描述了标架变换对 k 的依赖性。

In order to illustrate the working of the minimal essential scheme, we return to the example of a scalar field theory. Explicitly, we set

为了说明最小本质方案的工作原理，我们回到标量场论的例子。明确地，我们设

$$\Gamma_k[\chi] = \int d^dx \left\{\frac{Z_k}{2}\chi\left[-\partial^2 + m_k^2\right]\chi + \frac{Z_k^2\lambda_k}{12}\chi^4 + \cdots\right\}. \tag{81}$$

Here m_k and λ_k are scale-dependent couplings, Z_k is the wave function of the field, and the dots symbolize additional interaction terms. The wave-function renormalization constitutes an inessential coupling and we seek to remove it by a k - dependent frame transformation. Inspecting (81) indicates that this can be achieved by a k -dependent frame transformation which is linear in the field

此处 m_k 和 λ_k 是依赖尺度的耦合，Z_k 是场的波函数，省略号代表额外的相互作用项。波函数重整化是一种非本质耦合，我们希望通过依赖 k 的标架变换将其移除。分析 (81) 可知，这可以通过一个对场线性的、依赖 k 的标架变换实现

$$\phi_k = Z_k^{1/2}\chi \tag{82}$$

The kernel (80) then evaluates to $\Psi_k[\phi] = -\frac{1}{2}\eta_k\phi_k$ where $\eta_k \equiv -\partial_t \ln Z_k$ is the anomalous dimension of the field. Evaluating (79) then yields

核 (80) 由此变为 $\Psi_k[\phi] = -\frac{1}{2}\eta_k\phi_k$，其中 $\eta_k \equiv -\partial_t \ln Z_k$ 是场的反常维数。代入 (79) 计算后得到

$$\left(\partial_t - \frac{1}{2}\eta_k\phi\frac{\delta}{\delta\phi}\right)\Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)}[\phi] + \mathcal{R}_k\right)^{-1}(\partial_t\mathcal{R}_k - \eta_k\mathcal{R}_k)\right]. \tag{83}$$

The new functional $\Gamma_k[\phi]$ is then independent of Z_k. More precisely, the inessential coupling has been fixed to $Z_k = 1$ at all scales. The result (83) furthermore shows that η_k depends on the essential couplings of the theory only.

新泛函 $\Gamma_k[\phi]$ 随后就不依赖 Z_k 了。更准确地说，非本质耦合已经在所有尺度下被固定为 $Z_k = 1$。结果 (83) 进一步表明 η_k 仅依赖于理论的本质耦合。

As pointed out in [74] and illustrated by our explicit example above, the use of the frame-covariant flow equation in combination with the minimal-essential scheme may lead to significant technical simplifications when constructing solutions to the flow equation. In practice, these simplifications can be exploited systematically by parameterizing the kernel $\Psi_k[\phi]$ in terms of k-dependent γ-functions [78,79]. The freedom gained in this way can then be used to fix the inessential coupling constants to specific values. The scale dependence of the theory is then captured by the β-functions (governing the k-dependence of the essential couplings) and the γ-functions (governing the k-dependence of the inessential ones). Both sets of equations depend on the essential couplings only. The last property then simplifies the search for RG fixed points in a significant way.

正如文献 [74] 指出并由我们上述显式例子说明的那样，结合使用标架协变流方程与最小本质方案，在构造流方程的解时可以带来显著的技术简化。在实际计算中，通过用依赖 k 的 γ 函数参数化核 $\Psi_k[\phi]$，可以系统地利用这些简化 [78,79]。由此得到的自由度可以用来将非本质耦合常数固定为特定值。此时理论的标度依赖性由 β 函数 (描述本质耦合对 k 的依赖性) 和 γ 函数 (描述非本质耦合对 k 的依赖性) 共同刻画。两组方程都仅依赖本质耦合。这一性质能大幅简化重整化群不动点的搜索工作。

Flows in Terms of N-Type Cutoffs

N 型截断的流方程

Recently, a novel regularization scheme via dimensionless N-type cutoffs has been introduced [75, 76, 80], which may constitute a more physical alternative to the usually employed dimensionful UV cutoffs. The motivation for the introduction of a scale-free regularization scheme is the construction of regularized quantum systems, which have the potential of being physically realizable themselves. In this way, physical properties of the theory, which conventionally are to be studied in the quantum field theory limit, could already be probed at the level of the regularized system. Moreover, this scale-free regularization scheme is designed in a way such that self-consistent background geometries can easily be accessed.

最近，研究者提出了一种通过无量纲 N 型截断实现的新正则化方案 [75, 76, 80]，相较于常规使用的量纲化紫外截断，它可能是更符合物理实际的替代方案。引入无标度正则化方案的动机是构造正则化量子系统，这类系统本身就有可能在物理中实现。通过这种方式，理论中那些按惯例需要在量子场论极限下研究的物理性质，在正则化系统层面就可以被探测。此外，这种无标度正则化方案的设计使得自洽背景几何可以很容易地得到。

Schematically, the N -type cutoff regularizes the path integral (1) as follows. One expands the field in the eigenbasis of a suitable self-adjoint operator, e.g., the background Laplacian, such that the corresponding eigenvalues increase with $n \in \mathbb{N}$ (or $n \in \mathbb{R}^+$); cf. Eq. (47). Then the path integral is regularized by restricting the domain of integration to the field modes h^n with $n \leq N$. As a result, one obtains N -sequences of regularized quantum systems, which in principle are physically realizable.

概略来说，N 型截断对路径积分 (1) 的正则化过程如下: 将场展开到合适自伴算子 (例如背景拉普拉斯算子) 的本征基下，对应本征值会随 $n \in \mathbb{N}$ (或 $n \in \mathbb{R}^+$) 增大；参见式 (47)。随后通过将积分域限制到满足 $n \leq N$ 的场模式 h^n 来正则化路径积分，最终得到一系列原则上可物理实现的 N 正则化量子系统。

As a first application, the self-consistent spherical background geometries stemming from summing up the vacuum energy of a scalar field [75] as well as metric fluctuations [76] have been studied. The striking result, which is due to background independence, is that the self-consistent scalar curvatures $R(N)$ vanished for $N \to \infty$ in both cases. This is precisely the opposite behavior of the commonly perceived cosmological constant problem, according to which the background curvature, and therewith the total cosmological constant, should diverge when removing the UV regulator. Another striking result of this regularization scheme [75] is that N -type cutoffs give an explanation of the microscopical degrees of freedom which the Bekenstein-Hawking entropy of de Sitter space counts.

作为首个应用，该方案已被用于研究标量场真空能求和 [75] 以及度规涨落 [76] 得到的自洽球对称背景几何。由于背景无关性，一个引人注目的结果是，两种情况下自洽标量曲率 $R(N)$ 在 $N \to \infty$ 时都为零。这与常见认知中的宇宙学常数问题行为完全相反: 传统观点认为，移走紫外调节器时，背景曲率以及由此得到的总宇宙学常数应当发散。该正则化方案的另一个显著结果 [75] 是，N 型截断可以解释德西特空间贝肯斯坦-霍金熵计数的微观自由度。

The Einstein-Hilbert Truncation

爱因斯坦-希尔伯特截断

We proceed by giving an explicit example, illustrating how the Wetterich equation (44) is used to extract non-perturbative information about the gravitational RG flow. The discussion is based on the arguably simplest approximation for the effective average action Γ_k , the Einstein-Hilbert truncation. Starting from the seminal paper [11], this projection has been studied in detail in a series of works [81-85]. It still forms an integral part of studying the RG flow in many gravity-matter systems. The present exposition differs from the historical computations where the background metric has been set to the one of the maximally symmetric d -sphere S^d . Instead, we combine the idea of the universal RG machine [86,87] with off-diagonal heat-kernel techniques [86, 88-91] and carry out the derivation of the beta functions without specifying the background metric $\bar{g}_{\mu\nu}$. This stresses the background-independent nature of the computation and emphasizes the modern viewpoint on evaluating the FRGE in the context of gravity. In order to keep technical complications at the minimum, we adopt the harmonic gauge. The beta functions resulting from this setting are computed in section "Deriving the Beta Functions" and the resulting fixed point structure and phase diagram is presented in section "Fixed points, RG Trajectories, and Phase Diagram." Resorting to additional field decompositions, results generalizing the gauge fixing and regularization prescription have been obtained in [85] and corroborate the findings reviewed in this section.

我们接下来给出一个明确示例，说明如何利用 Wetterich 方程 (44) 提取引力重整化群流的非微扰信息。本讨论基于有效平均作用量 Γ_k 最简单可行的近似，即爱因斯坦-希尔伯特截断。自开创性论文 [11] 发表后，这一投影方法已经在一系列工作 [81-85] 中得到详细研究，至今仍是研究诸多引力-物质系统重整化群流不可或缺的一部分。本文的推导不同于以往将背景度规取为最大对称 d 球面 S^d 的历史计算；我们将通用重整化群机器的思想 [86,87] 与非对角热核技术 [86, 88-91] 结合，无需指定背景度规 $\bar{g}_{\mu\nu}$ 即可推导出贝塔函数。这体现了计算的背景无关性，也强调了引力框架下计算泛函重整化群方程的现代观点。为了尽可能降低技术复杂度，我们采用谐和规范。该设定下得到的贝塔函数在“推导贝塔函数”一节中计算，得到的不动点结构与相图在“不动点、重整化群轨迹与相图”一节中展示。借助额外的场分解，[85] 中得到了推广规范固定与正则化方案的结果，也证实了本节综述的结论。

Deriving the Beta Functions

推导 Beta 函数

The Einstein-Hilbert (EH) truncation works in the background approximation. Thus, the flow is obtained at zeroth order in the fluctuation fields. As a consequence only terms of zeroth and second order in the fluctuations are needed in the evaluation of the Wetterich equation. The projection of the flow equation tracks the scale dependence of the (background) Newton’s coupling G_k and the cosmological constant Λ_k . The gravitational part of the effective average action is approximated by the Einstein-Hilbert action

爱因斯坦-希尔伯特 (EH) 截断在背景近似下成立，因此流是在涨落场的零阶得到的。由此可知，计算 Wetterich 方程时仅需要涨落的零阶和二阶项。流方程的投影跟踪了 (背景) 牛顿耦合 G_k 和宇宙学常数 Λ_k 的标度依赖关系。有效平均作用量的引力部分由爱因斯坦-希尔伯特作用量近似

$$\Gamma_k^{\mathrm{EH}}[g] = \frac{1}{16\pi G_k} \int d^d x \sqrt{g} \left(-R + 2\Lambda_k\right), \tag{84}$$

with the couplings depending on the coarse-graining scale k . In view of the upcoming computation, it is convenient to introduce the dimensionless counterparts of Newton’s coupling and the cosmological constant as well as the anomalous dimension of Newton’s coupling

其中耦合依赖于粗粒化标度 k。为了后续计算，方便起见我们引入牛顿耦合、宇宙学常数的无量纲形式，以及牛顿耦合的反常维数

$$g_k \equiv k^{d-2}G_k,\ \lambda_k \equiv k^{-2}\Lambda_k,\ \eta_N(k) \equiv (G_k)^{-1}\partial_t G_k. \tag{85}$$

Furthermore, geometrical quantities constructed from $\bar{g}_{\mu\nu}$ are distinguished by a bar. For example, $\bar{D}_\mu$ is the covariant derivative constructed from the background metric.

此外，由 $\bar{g}_{\mu\nu}$ 构造的几何量会添加横杠加以区分，例如 $\bar{D}_\mu$ 是由背景度量构造的协变导数。

In order to obtain well-defined propagators, Γ_k^{EH} must be supplemented by a gauge-fixing term and the corresponding ghost action. Concretely, we implement a background gauge fixing

为了得到良定义的传播子，必须为 Γ_k^{EH} 补充规范固定项和相应的鬼作用量。具体来说，我们采用背景规范固定

$$\Gamma_k^{\mathrm{gf}}[h;\bar{g}] = \frac{1}{32\pi G_k\alpha}\int d^dx\sqrt{\bar{g}}\bar{g}^{\mu\upsilon}F_\mu F_\upsilon, \tag{86}$$

where the gauge-fixing condition is taken to be linear in the fluctuation field

其中规范固定条件取为涨落场的线性形式

$$F_\mu[h;\bar{g}] = \left[\delta_\mu^\alpha\bar{D}^\beta - \beta\bar{g}^{\alpha\beta}\bar{D}_\mu\right]h_{\alpha\beta}. \tag{87}$$

Here α and β are two gauge parameters which can largely be chosen arbitrary [85]. The ghost action accompanying (86) is found in the standard way and reads

此处 α 和 β 是两个规范参数，大多可以任意选择 [85]。对应式 (86) 的鬼作用量可通过标准方法得到，形式为

$$S^{\mathrm{ghost}}\left[h,\bar{C},C;\bar{g}\right] = -\sqrt{2}\int d^dx\sqrt{\bar{g}}\bar{C}_\mu\mathcal{M}[g,\bar{g}]^\mu{}_\upsilon C^\upsilon, \tag{88}$$

with the Faddeev-Popov operator being

其中法捷耶夫-波波夫算符为

$$\mathcal{M}[g,\bar{g}]^\mu{}_\upsilon = \bar{g}^{\mu\rho}\bar{D}^\sigma\left(g_{\rho\upsilon}D_\sigma + g_{\sigma\upsilon}D_\rho\right) - 2\beta\bar{g}^{\rho\sigma}\bar{D}^\mu g_{\sigma\upsilon}D_\rho. \tag{89}$$

Landau-type gauge fixings correspond to the limit $\alpha \to 0$ (with $\beta = 1/d$ being a preferred choice implementing the geometric gauge). The harmonic gauge adopted in the present computation sets $\alpha = 1$ (Feynman-type gauge) and $\beta = 1/2$. This has the technical advantage that all derivatives appear in the form of the background Laplace operator $\Delta = -\bar{g}^{\mu\upsilon}\bar{D}_\mu\bar{D}_\upsilon$.

朗道型规范固定对应极限 $\alpha \to 0$ (其中 $\beta = 1/d$ 是实现几何规范的优选选择)。本文计算采用的谐规范设定 $\alpha = 1$ (费曼型规范) 且 $\beta = 1/2$ 。这一选择的技术优势是，所有导数都可以表示为背景拉普拉斯算符 $\Delta = -\bar{g}^{\mu\upsilon}\bar{D}_\mu\bar{D}_\upsilon$ 的形式。

For the background computation ahead, it suffices to know the ghost action to the second order in the fluctuation fields. Adopting harmonic gauge and evaluating $\mathcal{M}[g,\bar{g}]^{\mu}{}_{\upsilon}\big|_{g=\bar{g}}$ shows that the relevant contributions are captured by

对于接下来的背景计算，只需知道鬼作用量在涨落场二阶的结果即可。采用谐规范计算 $\mathcal{M}[g,\bar{g}]^{\mu}{}_{\upsilon}\big|_{g=\bar{g}}$ 后可知，相关贡献可由下式表示

$$S^{\text{ghost}}\left[h=0,\bar{C},C;\bar{g}\right] = \sqrt{2}\int d^dx\sqrt{\bar{g}}\bar{C}_\mu\left[\delta^\mu_\upsilon\Delta - \bar{R}^\mu{}_\upsilon\right]C^\upsilon. \tag{90}$$

Here, we used the commutator of two background-covariant derivatives evaluated on vectors in order to combine the last two terms in (89) into the background Ricci scalar $\bar{R}_{\mu\nu}$. The approximation for the effective average action then combines the $\overline{\Gamma}_k[g]$ given in (84) with the gauge-fixing term (86) and the ghost action (88)

此处我们利用了作用在矢量上的两个背景协变导数的对易关系，将式 (89) 的最后两项合并为背景里奇标量 $\bar{R}_{\mu\nu}$ 。于是有效平均作用量的近似形式由式 (84) 给出的 $\overline{\Gamma}_k[g]$ ，结合规范固定项 (86) 和鬼作用量 (88) 组合得到

$$\Gamma_k\left[h,\bar{C},C;\bar{g}\right] \simeq \Gamma_k^{\text{EH}}[g] + \Gamma_k^{\text{gf}}[h;\bar{g}] + S^{\text{ghost}}\left[h,C,\bar{C};\bar{g}\right]. \tag{91}$$

At this stage a comment on the projection prescription is in order. Substituting (91) into its left-hand side and setting $g = \bar{g}$ afterwards indicates that the scale dependence of G_k and Λ_k can be read off from the coefficients multiplying

在此有必要对投影方案做一点说明。将式 (91) 代入流方程左侧，之后令 $g = \bar{g}$ ，就可以从乘子系数中读出 G_k 和 Λ_k 的标度依赖关系

$$\mathcal{O}_0 = \int d^dx\sqrt{\bar{g}},\ \mathcal{O}_1 = \int d^dx\sqrt{\bar{g}}\bar{R}. \tag{92}$$

All other interaction monomials spanning the gravitational theory space do not contribute to the computation. This entails the following, profound consequence. Equation (92) corresponds to a derivative expansion truncated at first order in the spacetime curvature. Hence, all terms containing two or more curvature tensors are outside the subspace spanned by our approximation. Moreover, (92) does not contain derivatives of a curvature tensor. Hence, there is no need to track such terms in the present computation. These considerations allow to formulate projection rules, stating that

张成引力理论空间的所有其他相互作用单项式都对本次计算没有贡献。这带来一个深刻的结论: 式 (92) 对应时空曲率一阶截断的导数展开，因此所有包含两个及以上曲率张量的项都不在我们近似的子空间之外。此外，式 (92) 不包含曲率张量的导数项，因此本次计算不需要追踪这类项。基于这些考虑，我们可以得到如下投影规则:

$$\bar{D}_\mu \bar{R}_{\alpha\beta\gamma\delta} \simeq 0,\ O\left(\bar{R}^2\right) \simeq 0. \tag{93}$$

We stress that these rules should not be read as restrictions on $\bar{g}_{\mu\nu}$. They merely identify structures which do not contribute to the computation. As a corollary of these relations, we conclude that we can freely commute covariant derivatives and curvature tensors, since the commutators just produce terms orthogonal to the projection space spanned by (92).

我们强调，这些规则不应被理解为对 $\bar{g}_{\mu\nu}$ 的限制，它们只是指出了哪些结构对计算没有贡献。作为上述关系的推论，我们得到结论: 我们可以自由对易协变导数和曲率张量，因为对易子产生的项都正交于式 (92) 张成的投影空间。

The first step in evaluating the trace appearing within the FRGE (44) consists in expanding (91) to second order in the fluctuation fields. In the ghost sector the result is already given in (90). For the gravitational fluctuations, we expand

计算 FRGE(式 (44)) 中迹的第一步，是将式 (91) 按涨落场展开到二阶。鬼部分的结果已经由式 (90) 给出，对于引力涨落，我们展开为

$$\Gamma_k\left[\bar{g}+h, \bar{g}\right] = \Gamma_k\left[\bar{g}, \bar{g}\right] + O\left(h\right) + \Gamma_k^{\text{quad}}\left[h; \bar{g}\right] + O\left(h^3\right). \tag{94}$$

The relevant coefficient $\Gamma_k^{\text{quad}}\left[h; \bar{g}\right]$ is readily found using computer algebra packages like xAct [92] and has the form

利用 xAct[92] 这类计算机代数包可以很容易得到相关系数 $\Gamma_k^{\text{quad}}\left[h; \bar{g}\right]$，其形式为

$$\Gamma_k^{\text{quad}} = \frac{1}{32\pi G_k}\int d^d x\sqrt{\bar{g}}\frac{1}{2}h_{\mu\nu}\left[K^{\mu\nu}{}_{\alpha\beta}\left(\Delta - 2\Lambda_k\right) + V^{\mu\nu}{}_{\alpha\beta}\right]h^{\alpha\beta}. \tag{95}$$

Here the ”kinetic” and ”potential” parts have the explicit form

此处“动能项”与“势项”有如下显式形式

$$K^{\mu\upsilon}{}_{\alpha\beta} = \frac{1}{2}\left(\delta^\mu_\alpha \delta^\upsilon_\beta + \delta^\mu_\beta \delta^\upsilon_\alpha - \bar{g}^{\mu\upsilon}\bar{g}_{\alpha\beta}\right), \tag{96}$$

$$V^{\mu\upsilon}{}_{\alpha\beta} = \bar{R}K^{\mu\upsilon}{}_{\alpha\beta} + \left(\bar{g}^{\mu\upsilon}\bar{R}_{\alpha\beta} + \bar{R}^{\mu\upsilon}\bar{g}_{\alpha\beta}\right) - 2\delta^{(\mu}_{(\alpha}\bar{R}^{\upsilon)}_{\beta)} - 2\bar{R}^{(\mu}{}_{(\alpha}{}^{\upsilon)}{}_{\beta)}.$$

The potential V collects all terms containing the spacetime curvature and is of first order in a curvature expansion.

势 V 囊括了所有包含时空曲率的项，在曲率展开中是一阶项。

In the next step, we would like to diagonalize the kinetic terms in the quadratic form (95). This can be achieved by decomposing $h_{\mu\nu}$ into component fields, resorting to the transverse-traceless decomposition [83, 93]. In the present case, it suffices to split the fluctuations into their trace and traceless part

下一步，我们希望将二次型 (95) 中的动能项对角化。这可以通过将 $h_{\mu\nu}$ 分解为分量场、利用横向无迹分解 [83, 93] 实现。在本文的情形中，只需将涨落分解为迹部分和无迹部分

$$h_{\mu\upsilon} = \widehat{h}_{\mu\upsilon} + \frac{1}{d}\bar{g}_{\mu\upsilon}h,\ \bar{g}^{\mu\upsilon}\widehat{h}_{\mu\upsilon} = 0. \tag{97}$$

Substituting this decomposition into (95) then yields

将该分解代入 (95) 后可得

$$\begin{aligned}\Gamma_k^{\text{quad}}[h;\bar{g}] = \frac{1}{32\pi G_k}\int d^dx\sqrt{\bar{g}}\Big[\frac{1}{2}\widehat{h}_{\mu\upsilon}\left[\Delta - 2\Lambda_k + \bar{R}\right]\widehat{h}^{\mu\upsilon} \\ -\left(\frac{d-2}{4d}\right)h\left[\Delta - 2\Lambda_k + \frac{d-4}{d}\bar{R}\right]h \\ -\bar{R}_{\mu\upsilon}\widehat{h}^{\mu\alpha}\widehat{h}_{\alpha}{}^{\upsilon} - \bar{R}_{\mu\upsilon\alpha\beta}\widehat{h}^{\mu\alpha}\widehat{h}^{\upsilon\beta} + \frac{d-4}{d}\bar{R}_{\mu\upsilon}h\widehat{h}^{\mu\upsilon}\Big].\end{aligned}$$

(98)

At this point, we are ready to specify the explicit form of the regulator $\mathcal{R}_k$. We dress up the Laplacians according to

至此，我们已经可以给出正规化项 $\mathcal{R}_k$ 的显式形式了。我们对拉普拉斯算子按如下方式修正

$$\Delta \mapsto \Delta + R_k \tag{99}$$

where $R_k(\Delta) = k^2R^{(0)}\left(\Delta/k^2\right)$ is the dimensionful cutoff function and $R^{(0)}(z)$ the corresponding profile. In the nomenclature of the review [27] this corresponds to a cutoff of type I. This choice implements the initial idea of supplying the fluctuation field with a k -dependent mass term. The resulting $\mathcal{R}_k$ is then diagonal in field space with its matrix elements given by

其中 $R_k(\Delta) = k^2R^{(0)}\left(\Delta/k^2\right)$ 是量纲 cutoff 函数，$R^{(0)}(z)$ 是对应轮廓。在综述文献 [27] 的命名规则中，这对应 I 型 cutoff。该选择实现了给涨落场添加依赖于 k 的质量项的初始思路。由此得到的 $\mathcal{R}_k$ 在场空间中是对角的，其矩阵元为

$$\mathcal{R}_k^{\widehat{h}\widehat{h}} = \frac{1}{32\pi G_k}R_k\mathbb{1}_{2T},\ \mathcal{R}_k^{hh} = -\frac{1}{32\pi G_k}\left(\frac{d-2}{2d}\right)R_k,\ \mathcal{R}_k^{\bar{C}C} = \sqrt{2}R_k\mathbb{1}_1.$$

(100)

Here

此处

$$\mathbb{1}_{2T\alpha\beta}^{\mu\upsilon} = \frac{1}{2}\left(\delta_\alpha^\mu\delta_\beta^\upsilon + \delta_\beta^\mu\delta_\alpha^\upsilon\right) - \frac{1}{d}\bar{g}^{\mu\upsilon}\bar{g}_{\alpha\beta},\ \mathbb{1}_{1\ \upsilon}^{\ \mu} = \delta_\upsilon^\mu, \tag{101}$$

are the units on the space of symmetric traceless two tensors (2T) and vectors (1), respectively.

分别是对称无迹二阶张量 (2T) 空间和向量 (1) 空间的单位。

We now proceed by constructing the inverse of the regularized Hessian. For the gravitational degrees of freedom, we encounter the two-by-two matrix

现在我们继续构造正规化黑塞矩阵的逆。对于引力自由度，我们得到如下二阶矩阵

$$\left[\Gamma_k^{(2)} + \mathcal{R}_k\right]^{ij} = \begin{bmatrix} K_{2T}(\Delta)\,\mathbb{1}_{2T} + V_{2T} & V_\times \\ V_\times^\dagger & K_0(\Delta)\,\mathbb{1}_0 + V_0 \end{bmatrix}. \tag{102}$$

Here $i, j = \{\widehat{h}, h\}$ labels the fields and we suppress all spacetime indices for the sake of readability. The explicit form of the kinetic functions K and the potentials V can be read off from Eqs. (98) and (90) and read

此处 $i, j = \{\widehat{h}, h\}$ 标记场，为了可读性我们省略了所有时空指标。动能函数 K 和势 V 的显式形式可以从式 (98) 和 (90) 中得到，即

$$\begin{aligned} K_{2T}(\Delta) &= \frac{1}{32\pi G_k}(\Delta + R_k - 2\Lambda_k) \\ K_0(\Delta) &= -\frac{1}{32\pi G_k}\left(\frac{d-2}{2d}\right)(\Delta + R_k - 2\Lambda_k), \\ K_1(\Delta) &= \sqrt{2}(\Delta R_k), \end{aligned} \tag{103}$$

and

和

$$\begin{aligned} V_{2T}{}^{\mu\upsilon}{}_{\alpha\beta} &= \frac{1}{32\pi G_k}\left(\bar{R}\mathbb{1}_{2T}{}^{\mu\upsilon}{}_{\alpha\beta} - 2\bar{R}_{(\alpha}^{(\mu}\delta_{\beta)}^{\upsilon)} - 2\bar{R}_{(\alpha}{}^{(\mu}{}_{\beta)}{}^{\upsilon)}\right), \\ V_0 &= -\frac{1}{32\pi G_k}\left(\frac{d-2}{2d}\right)\left(\frac{d-4}{d}\right)\bar{R}, \\ V_{\times\mu\nu} &= \frac{1}{32\pi G_k}\left(\frac{d-4}{d}\right)\left(\bar{R}_{\mu\nu} - \frac{1}{d}\bar{g}_{\mu\nu}\bar{R}\right), \\ V_1{}^\mu{}_\upsilon &= -\sqrt{2}\bar{R}^\mu{}_\upsilon \end{aligned} \tag{104}$$

Constructing the inverse of (102) builds on the exact inversion formula for block matrices

构造 (102) 的逆需要用到分块矩阵的精确求逆公式

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} \left(A - BD^{-1}C\right)^{-1} & -A^{-1}B\left(D - CA^{-1}B\right)^{-1} \\ -D^{-1}C\left(A - BD^{-1}C\right)^{-1} & \left(D - CA^{-1}B\right)^{-1} \end{bmatrix}. \tag{105}$$

Since the potentials (104) contain at least one power of the spacetime curvature, each entry can be constructed as a power series in V . The projection prescription (93) then indicates that it is sufficient to retain the terms up to one power of V . This implies, in particular, that the off-diagonal terms $V_\times$ do not enter into the present computation since they start to contribute at second order in V only. Taking into account that the regulator $\partial_t \mathcal{R}_k$ is diagonal in field space, it is sufficient to consider the diagonal entries in (105). Explicitly, the corresponding inverses are given by

由于势 (104) 至少包含一次时空曲率，每个矩阵元都可以展开为关于 V 的幂级数。投影规则 (93) 表明，我们只需保留到 V 的一阶项。这尤其意味着，非对角项 $V_\times$ 不会进入本次计算，因为它们对结果的贡献起始于 V 的二阶。考虑到正规化项 $\partial_t \mathcal{R}_k$ 在场空间是对角的，我们只需考虑 (105) 中的对角元，对应的逆矩阵显式为

$$(32\pi G_k)^{-1}\left[\Gamma_k^{(2)} + \mathcal{R}_k\right]^{-1}_{\hat{h}\hat{h}} \simeq \frac{1}{K_{2T}} - \frac{1}{K_{2T}} V_{2T} \frac{1}{K_{2T}} + O\left(V^2\right), \tag{106}$$

$$(32\pi G_k)^{-1}\left[\Gamma_k^{(2)} + \mathcal{R}_k\right]^{-1}_{hh} \simeq \frac{1}{K_0} - \frac{1}{K_0} V_0 \frac{1}{K_0} + O\left(V^2\right).$$

Based on these preliminary considerations, we can now write down the projected flow equation

基于这些初步分析，我们现在可以写出投影后的流方程

$$\begin{aligned} \partial_t \Gamma_k = & \frac{1}{2}\mathrm{Tr}_{2T}\left[\frac{1}{K_{2T}}\partial_t \mathcal{R}_k^{\hat{h}\hat{h}}\right] - \frac{1}{2}\mathrm{Tr}_{2T}\left[\frac{1}{K_{2T}} V_{2T} \frac{1}{K_{2T}}\partial_t \mathcal{R}_k^{\hat{h}\hat{h}}\right] \\ & + \frac{1}{2}\mathrm{Tr}_0\left[\frac{1}{K_0}\partial_t \mathcal{R}_k^{hh}\right] - \frac{1}{2}\mathrm{Tr}_0\left[\frac{1}{K_0} V_0 \frac{1}{K_0}\partial_t \mathcal{R}_k^{hh}\right] \\ & - \mathrm{Tr}_1\left[\frac{1}{K_1}\partial_t \mathcal{R}_k^{\bar{C}C}\right] + \mathrm{Tr}_1\left[\frac{1}{K_1} V_1 \frac{1}{K_1}\partial_t \mathcal{R}_k^{\bar{C}C}\right]. \end{aligned} \tag{107}$$

Here the subscripts $s = \{2T, 0, 1\}$ indicate that the traces are over traceless, symmetric matrices, scalars, and vectors, respectively.

此处下标 $s = \{2T, 0, 1\}$ 分别表示迹对无迹对称矩阵、标量和向量求迹。

Structurally, the traces (107) can be separated in traces without and with operator insertion V . In order to evaluate the resulting expressions of the first type, we use the early-time expansion of the heat kernel [94]

从结构上看，迹 (107) 可拆分为不含算符插入的迹和含算符插入的迹 V 。为了计算第一类拆分后得到的表达式，我们使用热核的早时展开 [94]

$$\mathrm{Tr}_s\left[e^{-s\Delta}\right] = \frac{1}{(4\pi s)^{d/2}}\,\mathrm{tr}\left(\mathbb{1}_s\right)\int d^dx\sqrt{\bar{g}}\left(1 + \frac{1}{6}s\bar{R}\right) + O\left(\bar{R}^2\right). \tag{108}$$

The trace $\mathrm{tr}\left(\mathbb{1}_s\right)$ counts the number of independent field components in each sector, i.e.,

迹 $\mathrm{tr}\left(\mathbb{1}_s\right)$ 对每个扇区中的独立场分量数计数，即

$$\mathrm{tr}\left(\mathbb{1}_0\right) = 1,\ \mathrm{tr}\left(\mathbb{1}_1\right) = d,\ \mathrm{tr}\left(\mathbb{1}_{2T}\right) = \frac{1}{2}(d-1)(d+2). \tag{109}$$

The heat kernel (108) can readily be extended to traces including functions of the Laplacian $W(\Delta)$. Formally introducing the (inverse) Laplace transform $\widetilde{W}(s)$ through $W(z) = \int_0^\infty ds \widetilde{W}(s) e^{-sz}$, we write

热核 (108) 可以很容易地推广到包含拉普拉斯函数 $W(\Delta)$ 的迹。通过 $W(z) = \int_0^\infty ds \widetilde{W}(s) e^{-sz}$ 形式化引入 (逆) 拉普拉斯变换 $\widetilde{W}(s)$ ，我们写出

$$\mathrm{Tr}_s [W(\Delta)] = \int_0^\infty ds \widetilde{W}(s) \mathrm{Tr}_s \left[e^{-s\Delta} \right]. \tag{110}$$

Substituting the early-time expansion (108) then yields

代入早时展开式 (108) 后得到

$$\mathrm{Tr}_s [W(\Delta)] = \frac{1}{(4\pi)^{d/2}} \mathrm{tr}(\mathbb{1}_s) \int d^d x \sqrt{\bar{g}} \left(Q_{d/2}[W] + \frac{1}{6} Q_{d/2-1}[W] \bar{R} \right) + O\left(\bar{R}^2\right),$$

(111)

where the Q -functionals are defined by

其中 Q 泛函定义为

$$Q_n[W] \equiv \int_0^\infty ds s^{-n} \widetilde{W}(s). \tag{112}$$

These functionals can be rewritten in terms of the original function $W(z)$:

这些泛函可以用原函数 $W(z)$ 改写为:

$$Q_n[W] = \frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} W(z),\ n > 0, \tag{113}$$

$$Q_0[W] = W(0).$$

For $n < 0$ one can always choose an integer k such that $n + k > 0$. Integrating by parts, one then establishes that

对于 $n < 0$ ，我们总能找到整数 k 满足 $n + k > 0$ 。随后通过分部积分可以证明

$$Q_n[W] = \frac{(-1)^k}{\Gamma(n+k)} \int_0^\infty dz z^{n+k-1} W^{(k)}(z),\ n < 0,\ n + k > 0. \tag{114}$$

At this point we note that the functions $W(z)$ appearing in (107) have the generic form

在此我们注意到，(107) 中出现的函数 $W(z)$ 具有如下一般形式

$$W(z) = \frac{G_k}{(z + R_k + w)^p} \partial_t \left(\frac{1}{G_k} R_k \right). \tag{115}$$

In this case, it is then convenient to trade the dimensionful Q -functionals with the dimensionless threshold functions

在这种情况下，将量纲为 Q 的泛函替换为无量纲阈函数是方便的

$$\Phi_n^p(w) \equiv \frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} \frac{R^{(0)} - zR^{(0)'}(z)}{(z + R^{(0)} + w)^p}, \tag{116}$$

$$\widetilde{\Phi}_n^p(w) \equiv \frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} \frac{R^{(0)}(z)}{(z + R^{(0)}(z) + w)^p},$$

where $R^{(0)}(p^2/k^2)$ is the dimensionless profile function associated with the regulator $R_k(p^2) = k^2 R^{(0)}(p^2/k^2)$. It is then readily verified that

其中 $R^{(0)}(p^2/k^2)$ 是调节器 $R_k(p^2) = k^2 R^{(0)}(p^2/k^2)$ 对应的无量纲轮廓函数，容易验证

$$Q_n\left[\frac{G_k}{(z + R_k + w)^p} \partial_t \left(\frac{1}{G_k} R_k\right)\right] = k^{2(n-p+1)} \left(2\Phi_n^p(w/k^2) - \eta_N \widetilde{\Phi}_n^p(w/k^2)\right),$$

(117)

where the anomalous dimension η_N has been introduced in (85). For the traces in the ghost sector, the G_k -dependence in (115) is absent, so that the terms proportional to η_N do not appear in this sector.

其中反常维数 η_N 已在 (85) 中引入。对于鬼扇区的迹，(115) 中的 G_k 依赖不存在，因此正比于 η_N 的项不会出现在该扇区中。

Starting from (107) the traces without potential insertions are readily evaluated by combining Eq. (111) with the result for the Q-functionals (117). The traces including the insertion of a potential can be evaluated along the same lines. Formally, such traces can be evaluated using the off-diagonal heat-kernel formulas provided in [86], since the potentials (104) do not contain any covariant derivatives and, owing to the projection prescription (93), can be treated as covariantly constant leading to significant simplifications though. In this case the relevant contributions are given by the leading term in the early-time expansion (108) with $\mathrm{tr}[\mathbb{1}_s] \to \mathrm{tr}[V_s]$. A brief computation establishes that

从 (107) 出发，结合 (111) 式与 Q 泛函的结果 (117)，即可计算不含势插入的迹；含势插入的迹可以沿相同路线计算。形式上，这类迹可以利用 [86] 中给出的非对角热核公式计算，因为势 (104) 不含任何协变导数，并且根据投影规则 (93)，势可以被处理为协变常数，这会带来显著的简化。在这种情况下，相关贡献由早时展开 (108) 中 $\mathrm{tr}[\mathbb{1}_s] \to \mathrm{tr}[V_s]$ 对应的领头项给出。经简单计算可得

$$\mathrm{tr}_1[V_1] = -\sqrt{2}\bar{R}$$

$$\mathrm{tr}_{2T}[V_{2T}] = \frac{1}{32\pi G_k} \frac{(d+2)(d^2 - 3d + 4)}{2d} \bar{R}. \tag{118}$$

The remaining terms in (107) are then found by pulling the contributions (118) out of the operator trace and evaluating the latter by again combining Eqs. (111) and (117). In this way one obtains the explicit form of the right-hand side of (107). Reading off the coefficients multiplying the interaction monomials (92) gives the equations governing the scale dependence of the dimensionful couplings G_k and Λ_k .

随后，我们将贡献 (118) 从算符迹中提出，再次结合 (111) 与 (117) 式计算算符迹，就得到了 (107) 中剩余的项。由此我们得到了 (107) 右侧的显式形式。读出相互作用单项式 (92) 的系数，就得到了量纲耦合 G_k 与 Λ_k 的标度演化满足的方程。

In order to study the renormalization group fixed points of the system, it is then natural to convert the dimensionful couplings to their dimensionless counterparts (85). The scale dependence of g_k and λ_k is encoded in the beta functions

为了研究系统的重整化群不动点，很自然地我们会将量纲耦合转化为对应的无量纲形式 (85)。g_k 和 λ_k 的标度依赖性由 beta 函数编码

$$\partial_t g_k = \beta_g\left(g_k, \lambda_k\right), \; \partial_t \lambda_k = \beta_\lambda\left(g_k, \lambda_k\right). \tag{119}$$

The explicit computation yields [11]

显式计算给出 [11]

$$\begin{aligned}
\beta_g &= \left(d-2+\eta_N\right) g \\
\beta_\lambda &= -\left(2-\eta_N\right)\lambda + \frac{g}{2(4\pi)^{d/2-1}}\left(2d\left(d+1\right)\Phi^1_{d/2}\left(-2\lambda\right)\right. \\
&\quad \left. -8d\Phi^1_{d/2}\left(0\right) - d\left(d+1\right)\eta_N\widetilde{\Phi}^1_{d/2}\left(-2\lambda\right)\right) .
\end{aligned} \tag{120}$$

The anomalous dimension η_N takes the form

反常维数 η_N 的形式为

$$\eta_N\left(g, \lambda\right) = \frac{gB_1\left(\lambda\right)}{1-gB_2\left(\lambda\right)}, \tag{121}$$

with

对于

$$\begin{aligned}
B_1\left(\lambda\right) &= \frac{1}{3}(4\pi)^{1-d/2}\left(d\left(d+1\right)\Phi^1_{d/2-1}\left(-2\lambda\right) - 6d\left(d-1\right)\Phi^2_{d/2}\left(-2\lambda\right)\right. \\
&\quad \left. -4d\Phi^1_{d/2-1}\left(0\right) - 24\Phi^2_{d/2}\left(0\right)\right) \\
B_2\left(\lambda\right) &= -\frac{1}{6}(4\pi)^{1-d/2}\left(d\left(d+1\right)\widetilde{\Phi}^1_{d/2-1}\left(-2\lambda\right) - 6d\left(d-1\right)\widetilde{\Phi}^2_{d/2}\left(-2\lambda\right)\right).
\end{aligned}$$

(122)

At this point we have completed the explicit derivation of the beta functions (120) governing the scale dependence of $\{g_k, \lambda_k\}$. The result agrees with the initial derivation [11], employing a maximally symmetric background. The present derivation shows, however, that this result is actually background independent. Apart from general properties related to the existence of the heat kernel (108), we never specified an explicit background $\bar{g}_{\mu\nu}$. Assuming its mere existence is sufficient to arrive at the final result.

至此我们已经完成了控制 $\{g_k, \lambda_k\}$ 标度依赖的 β 函数 (120) 的显式推导。该结果与最初采用极大对称背景得到的文献 [11] 推导一致。但本次推导表明，该结果实际上是不依赖背景的。除了与热核 (108) 存在性相关的一般性质外，我们从未指定具体的背景 $\bar{g}_{\mu\nu}$ ，仅假设其存在就足够得到最终结果。

Fixed points, RG Trajectories, and Phase Diagram

不动点、重整化群轨迹与相图

The beta functions (119) encode the dependence of Newton's coupling and the cosmological constant on the coarse-graining scale k . These have been derived for a generic regulator R_k . In order to investigate the resulting fixed point structure and phase diagram, we specify the regulator to be of Litim type (31). In this case, the integrals appearing in the threshold functions (116) can be evaluated analytically, yielding

β 函数 (119) 编码了牛顿耦合和宇宙常数对粗粒化尺度 k 的依赖关系，这些是针对一般调节函数 R_k 推导得到的。为了研究由此得到的不动点结构和相图，我们指定调节函数为 Litim 类型 (31)。在此情况下，阈函数 (116) 中出现的积分可以解析计算，得到

$$\Phi^p_n(w)^{\text{Litim}} = \frac{1}{\Gamma(n+1)} \frac{1}{(1+w)^p}, \quad \tilde{\Phi}^p_n(w)^{\text{Litim}} = \frac{1}{\Gamma(n+2)} \frac{1}{(1+w)^p}.$$

(123)

Upon substituting these expressions, the flow of $\{g_k, \lambda_k\}$ is governed by the coupled, nonlinear, autonomous, first-order differential equations (119) with

代入这些表达式后，$\{g_k, \lambda_k\}$ 的流由下述耦合非线性自治一阶微分方程 (119) 描述

$$\beta_g = (2 + \eta_N)\, g$$

$$\beta_\lambda = -(2 - \eta_N)\, \lambda + \frac{g}{8\pi} \left(\frac{20}{1-2\lambda} - 16 - \frac{5}{3} \eta_N \frac{1}{1-2\lambda} \right) \tag{124}$$

and

和

$$\eta_N = \frac{g \left(\frac{5}{1-2\lambda} - \frac{9}{(1-2\lambda)^2} - 7 \right)}{3\pi \left(1 + \frac{g}{12\pi} \left(\frac{5}{1-2\lambda} - \frac{6}{(1-2\lambda)^2} \right) \right)}. \tag{125}$$

Here we have specified $d = 4$ for explicitness.

此处我们为明确起见指定了 $d = 4$

We then determine the fixed points of this system. Since the truncation retains a finite number of interaction monomials $\mathcal{O}_i$ only, this search turns into the algebraic problem of finding the roots of the system $\left\{\beta_\lambda = 0, \beta_g = 0\right\}$; cf. Table 2. Inspecting (124), one finds two fixed points

接下来我们确定该系统的不动点。由于该截断仅保留有限个相互作用单项式 $\mathcal{O}_i$，寻找不动点转化为求解系统 $\left\{\beta_\lambda = 0, \beta_g = 0\right\}$ 根的代数问题；参见表 2。检查 (124) 式可以发现两个不动点

$$\text{GFP:} \;\; \{g_* = 0, \lambda_* = 0\},$$

(126)

$$\text{NGFP:} \; \{g_* = 0.707, \lambda_* = 0.193\}\;.$$

These correspond to a free and interacting theory, respectively. The NGFP is the projection of the Reuter fixed point onto the subspace spanned by the ansatz (84).

它们分别对应自由理论和相互作用理论。非高斯不动点 (NGFP) 是罗伊特不动点在由假设 (84) 张成子空间上的投影。

The stability properties of the RG flow in the vicinity of these fixed point are readily obtained by evaluating the stability matrix (10) for the beta functions (124). This yields

我们可以通过对 β 函数 (124) 计算稳定性矩阵 (10)，直接得到这些不动点附近重整化群流的稳定性性质，结果为

$$\text{GFP:} \;\; \{\theta_1 = 2, \theta_2 = -2\}, \tag{127}$$

$$\text{NGFP:} \;\; \left\{\theta_{1,2} = 1.48 \pm 3.04i\right\}\;.$$

Thus, the GFP constitutes a saddle point with one UV-attractive and one UV-repulsive eigendirection. Analyzing the corresponding eigenvectors shows that RG trajectories with a nonvanishing Newton's coupling are repelled by the GFP as $k \to \infty$. Hence, this fixed point cannot act as the UV completion of gravity. In contrast, the NGFP is UV-attractive for both g_k and λ_k. The complex stability coefficients indicate that the RG flow spirals into the fixed point as $k \to \infty$. Thus, this fixed point acts as UV completion for the RG trajectories entering its vicinity.

因此，高斯不动点 (GFP) 是一个鞍点，拥有一个紫外吸引本征方向和一个紫外排斥本征方向。对对应本征矢量的分析表明，当 $k \to \infty$ 时，牛顿耦合非零的重整化群轨迹会被 GFP 排斥，因此该不动点无法作为引力的紫外完备化。与之相对，NGFP 对 g_k 和 λ_k 都是紫外吸引的。复稳定性系数表明，当 $k \to \infty$ 时重整化群流会螺旋进入该不动点，因此这个不动点可以为进入其邻域的重整化群轨迹提供紫外完备化。

Treating the dimension d as a continuous parameter, one can trace the properties of the NGFP when performing an analytic continuation of the spacetime dimension. The results are summarized in Table 3. The table in the left panel gives the position and stability coefficients of the NGFP for selected values d , while the diagram in the right panel shows the position $g_*(d)$. The latter illustrates that the family of NGFPs emerges from the GFP in $d = 2 + \varepsilon$ dimensions. It can then be analytically continued up to $d = 4$. Thus, the Reuter fixed point is the analytic continuation of the NGFP seen in the ε -expansion around the free theory at the lower critical dimension $d = 2$ [81, 82] . For $d > 4$, the system (124) suggests that the NGFP continues to exist for all dimensions $d > 2$ [84]. At $d \gtrsim 5$, the existence of the fixed point turns into a regulator-dependent statement though [82]. Hence, it is currently unclear if there is an upper critical dimension where the family of NGFPs ceases to exist.

将维数 d 视为连续参数，我们可以在对时空维数做解析延拓时追踪 NGFP 的性质，结果汇总在表 3 中。左表给出了若干选定 d 值下 NGFP 的位置和稳定性系数，右图则展示了位置 $g_*(d)$ 。结果表明，NGFP 族从 $d = 2 + \varepsilon$ 维的 GFP 中产生，之后可以解析延拓到直到 $d = 4$ 。因此，罗伊特不动点就是下临界维度 $d = 2$ [81, 82] 自由理论附近 ε 展开中得到的 NGFP 的解析延拓。对于 $d > 4$ ，系统 (124) 表明，对所有维数 $d > 2$ ，NGFP 都持续存在 [84]。不过在 $d \gtrsim 5$ 处，该不动点的存在性会依赖于调节函数 [82]，因此目前尚不清楚是否存在一个上临界维度，使得 NGFP 族在该处不再存在。

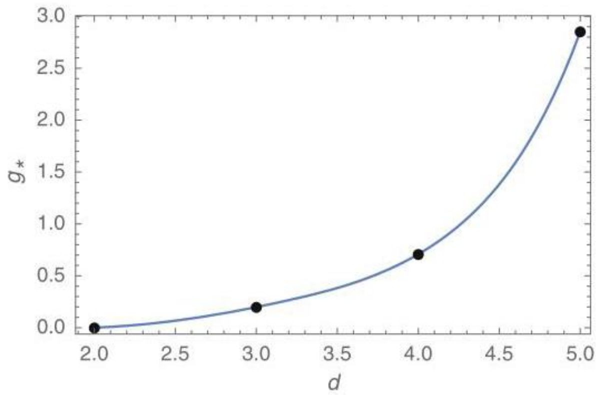
The system (124) is readily integrated numerically. The resulting phase diagram is governed by the interplay of the fixed points (126) and shown in Fig. 2. Here the magenta line indicates the position of a singular locus where η_N diverges. The physically relevant part of the phase diagram consists of the RG trajectories which emanate from the NGFP in the UV and cross over to the GFP as k decreases. A special role is thereby played by the separatrix (blue line) connecting the two fixed points. This trajectory leads to a vanishing cosmological constant $\lim_{k\to 0} \Lambda_k = 0$. The trajectories to the left of this line are classified as type Ia (orange line). Their characteristic feature is a negative cosmological constant, $\lim_{k\to 0} \Lambda_k < 0$. The trajectories to the right of the separatrix (green line) have been labeled type IIIa.

系统 (124) 很容易进行数值积分，得到的相图由不动点 (126) 的相互作用主导，如图 2 所示。图中品红色线指示了 η_N 发散的奇异轨迹位置。相图中物理相关的部分是从紫外区 NGFP 出发，在 k 减小时过渡到 GFP 的重整化群轨迹。其中连接两个不动点的分界线 (蓝色线) 发挥特殊作用，该轨迹最终得到宇宙常数 $\lim_{k\to 0} \Lambda_k = 0$ 为零。分界线左侧的轨迹被归类为 Ia 型 (橙色线)，其特征是宇宙常数为负，即 $\lim_{k\to 0} \Lambda_k < 0$ 。分界线右侧的轨迹 (绿色线) 被标记为 IIIa 型。

Table 3 Characteristics of the family of NGFPs in various dimensions d [81,82]. The table on the left gives the position and stability coefficients of the fixed points for selected values of d . The diagram to the right displays that the family emerges from the Gaussian fixed point in $d = 2 + \varepsilon$ and continuously connects to the Reuter fixed point in $d = 4$. Whether there is an upper critical dimension where the family of fixed points ceases to exist is currently an open question.

表 3 不同维度下 NGFP 族的性质 d [81,82]。左表给出了选定 d 取值下不动点的位置和稳定性系数。右图展示该不动点族由 $d = 2 + \varepsilon$ 维的高斯不动点产生，并连续连接到 $d = 4$ 维的罗伊特不动点。目前该不动点族是否存在一个不再存在的上临界维度仍是一个开放性问题。

	d	g_*	λ_*	θ_1	θ_2
GFP	d	0	0	$2-d$	2
NGFP	$2+\varepsilon$	$\frac{3}{38}\varepsilon$	$-\frac{3}{38}\varepsilon$	ε	$2+\frac{1}{19}\varepsilon$
NGFP	3	0.20	0.06		$1.15 \pm 0.83i$
NGFP	4	0.71	0.19		$1.48 \pm 3.04i$
NGFP	5	2.85	0.24		$2.69 \pm 5.15i$



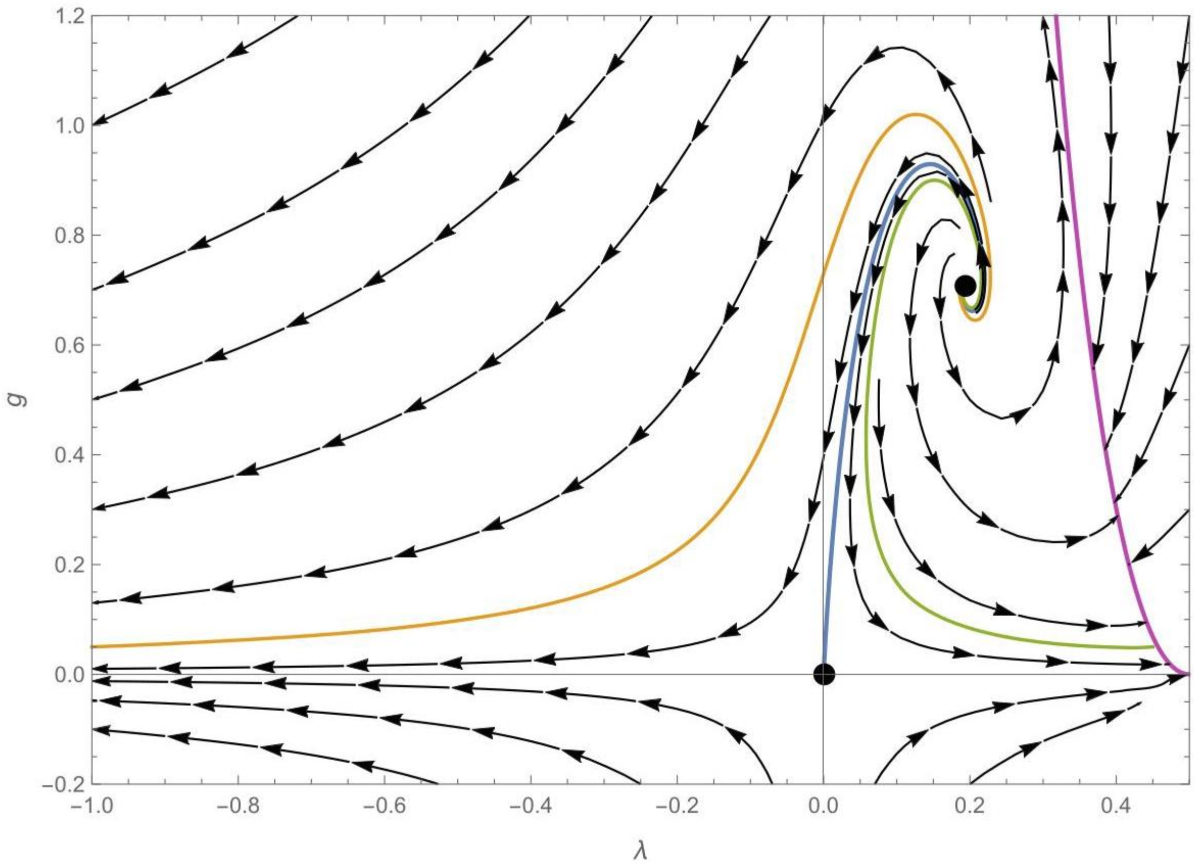


Fig. 2 Phase diagram constructed from integrating the beta functions (119) in $d=4$. All arrows point towards lower coarse-graining scale k. The GFP and NGFP are marked by the black dots, while the magenta line displays the locus where the anomalous dimension η_N diverges. The NGFP acts as a UV attractor capturing all trajectories in its vicinity. Lowering the coarse-graining scale, the flow undergoes a crossover towards

the GFP. The separatrix connecting the two fixed points is highlighted in blue. This solution leads to a vanishing cosmological constant $\Lambda_0 = 0$. Trajectories to the left of this line, exemplified by the orange trajectory, have been classified as type Ia and are characterized by $\Lambda_0 < 0$. RG trajectories to its right constitute the type IIIa solutions (represented by the green trajectory). They terminate at a finite value of k and lead to positive values $\Lambda_k > 0$. (Initially constructed in [82])

图 2 通过积分 $d = 4$ 中 β 函数 (119) 构造出的相图。所有箭头都指向更低的粗粒化标度 k 。GFP 和 NGFP 由黑点标记，品红色线显示反常维数 η_N 发散的轨迹。NGFP 作为紫外吸引子，吸引其邻域内的所有重整化群轨迹。降低粗粒化标度后，重整化群流会发生交叉转变，流向 GFP。连接两个不动点的分界线以蓝色高亮标出。该解对应宇宙学常数 $\Lambda_0 = 0$ 为零。这条线左侧的轨迹以橙色轨迹为例，被归类为 Ia 型，其特征为 $\Lambda_0 < 0$ 。它右侧的重整化群轨迹构成了 IIIa 型解 (由绿色轨迹表示)。这些轨迹终止于 k 的有限值，对应正的 $\Lambda_k > 0$ 。(最初构建于文献 [82])

They terminate at the singular locus at a finite value of k . In the vicinity of the GFP, they exhibit a regime where Λ_k is constant and positive. It is expected that nature is described by an RG trajectory within this class [95]. This trajectory is special in the sense that it almost hits the GFP. Only at the very last moment, it takes a turn flowing away from the fixed point. In this way the trajectory accommodates the tiny value of the cosmological constant observed in cosmology. Thus, the Einstein-Hilbert truncation does not predict the value of the cosmological constant. It is a function of the free parameters labeling the RG trajectories leaving the NGFP. The cosmological constant then has the role of an experimental input which is used to identify the RG trajectory realized in nature.

它们终止于奇异轨迹上 k 的有限值处。在 GFP 邻域内，它们存在一个 Λ_k 恒为正的区域。一般认为自然就是由这类重整化群轨迹描述的 [95]。这条轨迹的特殊之处在于它几乎命中 GFP，仅在最后一刻转向离开不动点。通过这种方式，该轨迹可以容纳宇宙学中观测到的极小宇宙学常数。因此，爱因斯坦-希尔伯特截断无法预言宇宙学常数的取值。宇宙学常数是标记离开 NGFP 的重整化群轨迹的自由参数的函数，其作用是作为实验输入，用以确定自然界中实现的具体重整化群轨迹。

At this stage, it is instructive to pick a generic RG trajectory of type IIIa and illustrate the k -dependence of the dimensionful couplings. The resulting flow of G_k and Λ_k is exemplified in Fig. 3 where all dimensionful quantities are given in units of the Planck scale $M_{\rm Pl} = G_0^{-1/2}$. For $k > 1$ the scale dependence is governed by the NGFP, while for $k < 1$ the flow is controlled by the GFP. As a result, the flow of the couplings interpolates between the scaling regimes

在这一步，选取一条典型的 IIIa 型重整化群轨迹，说明量纲耦合对 k 的依赖关系是很有启发意义的。由此得到的 G_k 和 Λ_k 的重整化群流如图 3 所示，图中所有量纲量都以普朗克能标 $M_{\rm Pl} = G_0^{-1/2}$ 为单位。对于 $k > 1$ ，标度依赖性由 NGFP 主导，而对于 $k < 1$ ，重整化群流由 GFP 控制。因此，耦合的流在不同标度区之间插值

$$\text{NGFP:} \;\; G_k \simeq g_* k^{-2}, \; \Lambda_k \simeq \lambda_* k^2 \;\; k > 1 \, , \tag{128}$$

$$\text{GFP:} \;\; G_k \simeq G_0, \; \Lambda_k \simeq \Lambda_0 \;\; k < 1 \, .$$

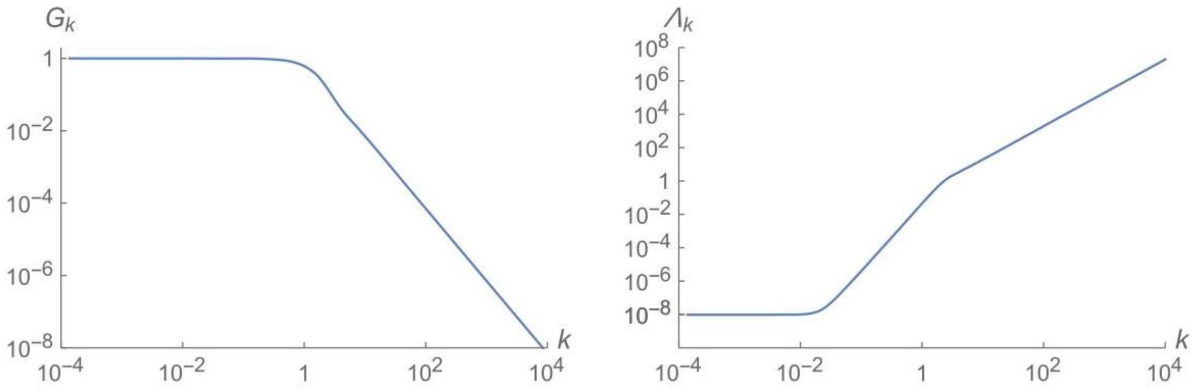


Fig. 3 Dependence of the dimensionful Newton's coupling (left panel) and cosmological constant (right panel) on the coarse-graining scale along a typical RG trajectory of type IIIa. The flow interpolates between the classical regime ($k \ll 1$) where G_k and Λ_k are constant and the fixed point regime ($k \gg 1$) where $G_k \propto k^{-2}$ and $\Lambda_k \propto k^2$. By definition, the crossover between these scaling regimes occurs at the Planck scale $M_{\rm Pl} \equiv G_0^{-1/2}$ which is generated dynamically when flowing away from the NGFP. Notably, Λ_k exhibits an intermediate scaling regime where $\Lambda_k \propto k^4$. All quantities are measured in units of $M_{\rm Pl}$. (Adaptation from [96])

图 3 沿典型 IIIa 型重整化群轨迹，量纲牛顿耦合 (左图) 和宇宙学常数 (右图) 对粗粒化标度的依赖关系。重整化群流在经典区 ($k \ll 1$) (G_k 和 Λ_k 在此区为常数) 和不动点区 ($k \gg 1$) ($G_k \propto k^{-2}$ 和 $\Lambda_k \propto k^2$ 在此区满足标度关系) 之间插值。根据定义，这些标度区之间的交叉转变发生在普朗克能标 $M_{\rm Pl} \equiv G_0^{-1/2}$ ，该能标是离开 NGFP 流动时动力学产生的。值得注意的是，Λ_k 存在一个中间标度区，$\Lambda_k \propto k^4$ 在该区域满足标度关系。所有物理量都以 $M_{\rm Pl}$ 为单位计量。(改编自文献 [96])

The crossover between the two regimes occurs at the Planck scale. This scale is generated dynamically when moving away from the NGFP. Classical general relativity (in the sense of a low-energy effective field theory) is then recovered in the vicinity of the GFP.

两个区域之间的交叉转变发生在普朗克能标，该能标是离开 NGFP 时动力学产生的。经典广义相对论 (作为低能有效场论) 就可以在 GFP 的邻域中被重建出来。

We conclude by stressing that the exact fixed point action Γ_* associated with the Reuter fixed point does (most likely) not coincide with the Einstein-Hilbert action. While this may be suggested by the analysis of this section, one has to account for the fact that we have been working within a projection of the full theory space to this two-dimensional subspace. Additional contributions to Γ_* , as, e.g., higher-derivative terms, are not visible in this analysis.

最后我们强调，与罗伊特不动点相关的精确不动点作用量 Γ_* (极有可能) 并不与爱因斯坦-希尔伯特作用量一致。尽管本节的分析可能会给人这样的暗示，但我们必须注意，我们的分析是在将全理论空间投影到该二维子空间内进行的。该分析无法体现 Γ_* 的额外贡献，例如高阶导数项这类贡献。

Further Reading

延伸阅读

The Einstein-Hilbert truncation discussed in this section constitutes the starting point for understanding the theory space of gravity, its RG fixed points, and their mutual relations. By now, this exploration has made significant progress in moving beyond this basic example. At the level of the background approximation, $f(R)$ - type projections have been studied in polynomial approximations to very high order [97-101] and it has been shown that the NGFP also persists once the two-loop counterterm identified by Goroff and Sagnotti [2, 3] is included in the projection [102]. Within the fluctuation approach, there has been significant progress on understanding the momentum structure of the graviton propagator [70, 71] and resolving the momentum dependence of three- and four-point vertices [72, 73] . In parallel, a program geared towards developing asymptotically safe amplitudes has been initiated in [40]. Covering these developments in detail is beyond the scope of this introductory chapter and the interested reader may consult the more advanced chapters of this volume for further information.

本节讨论的爱因斯坦-希尔伯特截断是理解引力理论空间、其重整化群不动点以及它们相互关系的起点。时至今日，这项探索在超越该基础示例的方向上已经取得了重大进展。在背景近似层面，多项式逼近中已研究了 $f(R)$ 型投影，将其拓展到了非常高的阶 [97-101]，并且研究表明，当投影中包含 Goroff 和 Sagnotti 确定的两圈 counterterm [2, 3] 后，非高斯不动点 (NGFP) 仍然存在 [102]。在涨落方法框架下，人们在理解引力子传播子的动量结构 [70, 71]，以及解析三顶点和四顶点的动量依赖关系 [72, 73] 方面取得了显著进展。与此同时，文献 [40] 已经启动了一个旨在发展渐近安全振幅的项目。详细介绍这些进展超出了这一入门章节的范围，感兴趣的读者可以查阅本卷中更进阶的章节获取更多信息。

Concluding Comments

总结评述

The Wetterich equation (2) constitutes an essential tool in developing the gravitational asymptotic safety program. Starting from its adaption to gravity [11], it has provided substantial evidence for the existence of a viable interacting renormalization group fixed point - the Reuter fixed point - which could provide a consistent and predictive high-energy completion of the gravitational interactions.

韦特里希方程 (2) 是发展引力渐近安全项目的核心工具。自从它被适配到引力领域 [11] 以来，已经为存在一个可行的相互作用重整化群不动点——罗伊特不动点——提供了大量证据，该不动点可以为引力相互作用提供一个自洽且具备预言性的高能完备化。

The present chapter focused on the case where the gravitational degrees of freedom are carried by the metric field. The applicability of the functional renormalization group and in particular the Wetterich equation is not limited to this setting though. It has readily been extended to other sets of fields which, at the classical level, encode the same gravitational dynamics as general relativity. Notably, this includes the case where the gravitational degrees of freedom are encoded in the vielbein (”tetraed only” formulation) [103, 104], the Palatini formalism [105-108], the Arnowitt-Deser-Misner decomposition [8, 109-112], and unimodular gravity

[113-116]. While the exploration of the corresponding theory spaces is far less developed than the one for the metric theory, there are indications that these settings also possess interacting renormalization group fixed points suitable for rendering the construction asymptotically safe. In the case of unimodular gravity, there are arguments that the theory is in the same universality class as the metric formulation [117]. Whether the other fixed points are in the universality class of the Reuter fixed point is an open question though.

本章聚焦于引力自由度由度规场承载的情形。然而，泛函重整化群，尤其是韦特里希方程的适用性并不局限于这一框架。它已经被顺利拓展至其他场体系，这些场在经典层面可以编码和广义相对论一致的引力动力学。值得注意的是，这其中包括引力自由度由标架编码的情形 (“仅四标架”表述)[103, 104]、帕拉蒂尼形式 [105-108]、阿尔诺维特-德泽-米斯纳分解 [8, 109-112] 以及幺模引力 [113-116]。尽管对应理论空间的探索远不及度规理论成熟，已有迹象表明这些框架同样存在适合让该构造实现渐近安全的相互作用重整化群不动点。对于幺模引力，有观点认为该理论与度规表述属于同一个普适类 [117]。不过其他不动点是否属于罗伊特不动点的普适类目前仍是开放性问题。

Notably, there also has been progress aiming at the implementation of the renormalization group on discrete geometries. In the context of the causal dynamical triangulation program [12, 13] , renormalization group flows have been constructed in [118]. The underlying idea is to pick an observable whose value is held constant when varying the parameters of the Monte Carlo simulation. This led to the surprising conclusion that the phase-transition line expected to provide the high-energy completion of the theory actually appears to be approached in the infrared.

值得注意的是，离散几何上实现重整化群的工作也已经取得了进展。在因果动态三角剖分框架 [12, 13] 中，重整化群流已在文献 [118] 中被构造出来。其核心思想是选取一个可观测量，在改变蒙特卡洛模拟参数时保持该可观测量的值不变。这得出了一个令人惊讶的结论: 原本被认为提供理论高能完备化的相变线，实际上似乎是在红外区域被趋近。

So far, our discussion has focused on gravitational degrees of freedom only. It is rather straightforward to extend this construction by including additional matter fields as well as all the building blocks of the standard model of particle physics [33,119]. Many of the gravity-matter systems investigated to date exhibit interacting renormalization group fixed points whose properties are very similar to the ones found for the Reuter fixed point. Since the beta functions encoding the fixed point structure of these systems depend on the number of matter fields in a continuous way, it is likely that the Reuter fixed point is part of a continuous web of interacting fixed points. Since it is unlikely that these encode the same universality class, it is suggestive to refer to these as deformed Reuter fixed points, highlighting that the systems actually realize different (albeit related) universal behaviors. A detailed summary of the state of the art in investigating asymptotically safe gravity-matter systems is beyond the scope of this elementary introduction and we refer to the recent reviews [33, 119] ”gravity-matter systems” in Asymptotically Safe gravity. In short, it is conceivable though that the asymptotic safety mechanism may lead to a unified theory incorporating the standard model of particle physics and gravity within the framework of a relativistic quantum field theory. This exciting perspective certainly warrants further investigation.

迄今为止，我们的讨论仅聚焦于引力自由度。将这一构造拓展，纳入额外物质场以及粒子物理标准模型的所有基础组成部分是十分直接的 [33, 119]。目前已研究的多数引力-物质系统都存在相互作用重整化群不动点，其性质与罗伊特不动点的性质非常相似。由于编码这些系统不动点结构的 β 函数会随物质场的数量连续变化，罗伊特不动点很可能是一张相互作用不动点连续网的一部分。由于这些不动点不太可能属于同一个普适类，将它们称为变形罗伊特不动点是合理的，这也体现了这些系统实际上实现了不同 (尽管彼此相关) 的普适行为。对渐近安全引力-物质系统研究的现有进展进行详细总结超出了这一基础介绍的范围，我们推荐读者参考近期综述 [33, 119]《渐近安全引力中的引力-物质系统》。简而言之，渐近安全机制很有可能在相对论量子场论框架内，得到一个统一了粒子物理标准模型和引力的理论。这一激动人心的前景无疑值得进一步研究。

Cross-References

交叉引用

Asymptotic Safety of Gravity with Matter

含物质引力的渐近安全

Acknowledgments My understanding of the functional renormalization group and its applications in the context of gravity has benefited enormously from countless discussions with many colleagues. It is therefore my pleasure to thank J. Ambjørn, D. Becker, M. Becker, W. Beenakker, A. Bonanno, L. Bosma, T. Budd, L. Buoninfante, J. Donoghue, T. Draper, A. Eichhorn, A. Ferreiro, R. Ferrero, G. Gubitosi, R. Kleiss, A. Koshelev, R. Loll, M. Niedermaier, R. Ooijer, C. Pagani, J. M. Pawlowski, R. Percacci, A. D. Pereira, S. Pirlo, A. Platania, M. Reichert, M. Schiffer, O. Zanusso, and C. Wetterich for sharing their insights and views. In addition, I want to thank B. Knorr and C. Ripken for their close collaboration on many aspects presented in this review. Finally, I would like to thank M. Reuter for introducing me to this subject and his continual advice and support.

致谢我对泛函重整化群及其在引力背景下应用的理解，极大受益于与众多同事的无数讨论。因此，我十分荣幸感谢 J. 安博恩、D. 贝克尔、M. 贝克尔、W. 比纳克、A. 博南诺、L. 博斯马、T. 巴德、L. 布宁凡特、J. 多诺霍、T. 德雷珀、A. 艾希霍恩、A. 费雷罗、R. 费雷罗、G. 古比托西、R. 克赖斯、A. 科舍列夫、R. 洛尔、M. 尼德迈尔、R. 奥耶尔、C. 帕加尼、J.M. 帕洛夫斯基、R. 佩尔卡奇、A.D. 佩雷拉、S. 皮洛、A. 普拉塔尼亚、M. 赖克特、M. 希弗、O. 扎努索和 C. 韦特里希分享他们的洞见与观点。此外，我要感谢 B. 克诺尔和 C. 里彭肯，他们在这篇综述所呈现的诸多内容中都与我开展了密切合作。最后，我要感谢 M. 罗伊特，是他将我引入这一研究领域，并一直给予我建议与支持。

References

参考文献

1. G. ’t Hooft, M.J.G. Veltman, One loop divergencies in the theory of gravitation. Ann. Poincare Phys. Theor. A20, 69-94 (1974)

2. M.H. Goroff, A. Sagnotti, Quantum gravity at two loops. Phys. Lett. B160, 81 (1985)

3. M.H. Goroff, A. Sagnotti, The ultraviolet behavior of Einstein gravity. Nucl. Phys. B266, 709 (1986)

4. A.E.M. van de Ven, Two loop quantum gravity. Nucl. Phys. B378, 309-366 (1992)

5. J. Armas, Conversations on Quantum Gravity (Cambridge University Press, Cambridge, 2021), p. 8

6. R. Loll, G. Fabiano, D. Frattulillo, F. Wagner, Quantum Gravity in 30 Questions. PoS CORFU2021 6, 316 (2022)

7. P. Hořava, Quantum gravity at a Lifshitz point. Phys. Rev. D 79, 084008 (2009)

8. S. Rechenberger, F. Saueressig, A functional renormalization group equation for foliated spacetimes. JHEP 03, 010 (2013)

9. S. Weinberg, Critical Phenomena for Field Theorists, in Erice Subnuclear Physics (Springer, New York, 1976), p. 1

10. S. Weinberg, Ultraviolet Divergences in Quantum Theories of Gravitation, in General Relativity: An Einstein centenary survey, eds. by S.W. Hawking, W. Israel (Cambridge University Press, Cambridge, 1979), pp. 790-831

11. M. Reuter, Nonperturbative evolution equation for quantum gravity. Phys. Rev. D57, 971-985 (1998)

12. J. Ambjørn, A. Görlich, J. Jurkiewicz, R. Loll, Nonperturbative quantum gravity. Phys. Rept. 519, 127-210 (2012)

13. R. Loll, Quantum gravity from causal dynamical triangulations: a review. Class. Quant. Grav. 37(1), 013002 (2020)

14. J. Ambjorn, L. Glaser, A. Goerlich, J. Jurkiewicz, Euclidian 4d quantum gravity with a nontrivial measure term. JHEP 10, 100 (2013)

15. D. Coumbe, J. Laiho, Exploring Euclidean dynamical triangulations with a non-trivial measure term. JHEP 04, 028 (2015)

16. T. Rindlisbacher, P. de Forcrand, Euclidean dynamical triangulation revisited: is the phase transition really 1st order? (extended version). JHEP 05, 138 (2015)

17. S. Bassler, J. Laiho, M. Schiffer, J. Unmuth-Yockey, The de sitter instanton from euclidean dynamical triangulations. Phys. Rev. D 103, 114504 (2021)

18. M. Asaduzzaman, S. Catterall, Euclidean dynamical triangulations revisited. Phys. Rev. D 107(7), 074505 (2023)

19. J. Ambjorn, S. Jordan, J. Jurkiewicz, R. Loll, A second-order phase transition in CDT. Phys. Rev. Lett. 107, 211303 (2011)

20. C. Wetterich, Exact evolution equation for the effective potential. Phys. Lett. B301, 90-94 (1993)

21. T.R. Morris, The exact renormalization group and approximate solutions. Int. J. Mod. Phys. A9, 2411-2450 (1994)

22. R. Percacci, An Introduction to Covariant Quantum Gravity and Asymptotic Safety, in 100 Years of General Relativity, vol. 3 (World Scientific, Singapore, 2017)

23. M. Reuter, F. Saueressig, Quantum Gravity and the Functional Renormalization Group (Cambridge University Press, Cambridge, 2019)

24. S. Nagy, Lectures on renormalization and asymptotic safety. Ann. Phys. 350, 310-346 (2014)

25. M. Reichert, Lecture notes: functional renormalisation group and asymptotically safe quantum gravity. PoS Modave2019, 005 (2020)

26. M. Niedermaier, M. Reuter, The asymptotic safety scenario in quantum gravity. Living Rev. Rel. 9, 5-173 (2006)

27. A. Codello, R. Percacci, C. Rahmede, Investigating the ultraviolet properties of gravity with a Wilsonian renormalization group equation. Ann. Phys. 324, 414-469 (2009)

28. M. Reuter, F. Saueressig, Quantum Einstein gravity. New J.Phys. 14, 055022 (2012)

29. J. Berges, N. Tetradis, C. Wetterich, Nonperturbative renormalization flow in quantum field theory and statistical physics. Phys. Rept. 363, 223-386 (2002)

30. H. Gies, Introduction to the functional RG and applications to gauge theories. Lect. Notes Phys. 852, 287-348 (2012)

31. J.M. Pawlowski, Aspects of the functional renormalisation group. Ann. Phys. 322, 2831-2915 (2007)

32. N. Dupuis, L. Canet, A. Eichhorn, W. Metzner, J.M. Pawlowski, M. Tissier, N. Wschebor, The non-perturbative functional renormalization group and its applications. Phys. Rep. 910, 1-114 (2021)

33. A. Eichhorn, An asymptotically safe guide to quantum gravity and matter. Front. Astron. Space Sci. 5, 47 (2019)

34. J.M. Pawlowski, M. Reichert, Quantum gravity: a fluctuating point of view. Front. Phys. 8, 551848 (2021)

35. B. Koch, F. Saueressig, Black holes within asymptotic safety. Int. J. Mod. Phys. A29(8), 1430011 (2014)

36. A. Bonanno, F. Saueressig, Asymptotically safe cosmology - a status report. Comp. Rendus Phys. 18, 254-264 (2017)

37. A. Bonanno, A. Eichhorn, H. Gies, J.M. Pawlowski, R. Percacci, M. Reuter, F. Saueressig, G. P. Vacca, Critical reflections on asymptotically safe gravity. Front. Phys. 8, 269 (2020)

38. R. Gastmans, R. Kallosh, C. Truffin, Quantum gravity near two-dimensions. Nucl. Phys. B 133, 417-434 (1978)

39. S.M. Christensen, M.J. Duff, Quantum gravity in two + ε dimensions. Phys. Lett. B 79, 213- 216 (1978)

40. T. Draper, B. Knorr, C. Ripken, F. Saueressig, Finite quantum gravity amplitudes: no strings attached. Phys. Rev. Lett. 125(18), 181301 (2020)

41. T. Draper, B. Knorr, C. Ripken, F. Saueressig, Graviton-mediated scattering amplitudes from the quantum effective action. JHEP 11, 136 (2020)

42. B. Knorr, C. Ripken, F. Saueressig, Form factors in quantum gravity - contrasting nonlocal, ghost-free gravity and asymptotic safety. Nuovo Cim. C 45(2), 28 (2022)

43. C. Wetterich, Quantum scale symmetry, vol. 1 (2019). arXiv:1901.04741

44. M. Rocek, R.M. Williams, Quantum Regge calculus. Phys. Lett. B 104, 31 (1981)

45. H.W. Hamber, Quantum gravity on the lattice. Gen. Rel. Grav. 41, 817-876 (2009)

46. M. Reuter, C. Wetterich, Effective average action for gauge theories and exact evolution equations. Nucl. Phys. B417, 181-214 (1994)

47. W. Houthoff, A. Kurov, F. Saueressig, On the scaling of composite operators in asymptotic safety. JHEP 04, 099 (2020)

48. A. Kurov, F. Saueressig, On characterizing the quantum geometry underlying asymptotic safety. Front. Phys. 8, 187 (2020)

49. J.F. Donoghue, A critique of the asymptotic safety program. Front. Phys. 8, 56 (2020)

50. K.G. Wilson, J.B. Kogut, The renormalization group and the epsilon expansion. Phys. Rept. 12, 75-199 (1974)

51. M. Reuter, Effective Average Actions and Nonperturbative Evolution Equations CORFU (1995)

52. J. Braun et al., Renormalised Spectral Flows, vol. 6 (2022). arXiv:2206.10232

53. E. Manrique, M. Reuter, Bare action and regularized functional integral of asymptotically safe quantum gravity. Phys. Rev. D79, 025008 (2009)

54. N. Ohta, R. Percacci, A.D. Pereira, Gauges and functional measures in quantum gravity I: Einstein theory. JHEP 06, 115 (2016)

55. N. Ohta, R. Percacci, A.D. Pereira, Gauges and functional measures in quantum gravity II: higher derivative gravity. Eur. Phys. J. C 77(9), 611 (2017)

56. M. Demmel, A. Nink, Connections and geodesics in the space of metrics. Phys. Rev. D92(10), 104013 (2015)

57. A. Nink, M. Reuter, The unitary conformal field theory behind 2D asymptotic safety. JHEP 02, 167 (2016)

58. D. Becker, M. Reuter, Towards a C -function in 4D quantum gravity. JHEP 03,065 (2015)

59. C. Pagani, M. Reuter, Background independent quantum field theory and gravitating vacuum fluctuations. Ann. Phys. 411, 167972 (2019)

60. E. Manrique, M. Reuter, Bimetric truncations for quantum Einstein gravity and asymptotic safety. Ann. Phys. 325, 785-815 (2010)

61. E. Manrique, M. Reuter, F. Saueressig, Bimetric renormalization group flows in quantum Einstein gravity. Ann. Phys. 326, 463-485 (2011)

62. E. Manrique, M. Reuter, F. Saueressig, Matter induced bimetric actions for gravity. Ann. Phys. 326, 440-462 (2011)

63. S.A. Fulling, R.C. King, B.G. Wybourne, C.J. Cummins, Normal forms for tensor polynomials. I: the Riemann tensor. Class. Quant. Grav. 9, 1151-1197 (1992)

64. Y. Decanini, A. Folacci, FKWC-Bases and Geometrical Identities for Classical and Quantum Field Theories in Curved Spacetime (2008). arXiv:0805.1595

65. D. Becker, C. Ripken, F. Saueressig, On avoiding Ostrogradski instabilities within asymptotic safety. JHEP 12, 121 (2017)

66. A. Platania, C. Wetterich, Non-perturbative unitarity and fictitious ghosts in quantum gravity. Phys. Lett. B 811, 135911 (2020)

67. A. Platania, Causality, unitarity and stability in quantum gravity: a non-perturbative perspective. JHEP 09, 167 (2022)

68. B. Knorr, C. Ripken, F. Saueressig, Form factors in asymptotic safety: conceptual ideas and computational toolbox. Class. Quant. Grav. 36(23), 234001 (2019)

69. L. Bosma, B. Knorr, F. Saueressig, Resolving spacetime singularities within asymptotic safety. Phys. Rev. Lett. 123(10), 101301 (2019)

70. N. Christiansen, B. Knorr, J.M. Pawlowski, A. Rodigast, Global flows in quantum gravity. Phys. Rev. D93(4), 044036 (2016)

71. A. Bonanno, T. Denz, J.M. Pawlowski, M. Reichert, Reconstructing the graviton. Sci. Post Phys. 12(1), 001 (2022)

72. N. Christiansen, B. Knorr, J. Meibohm, J.M. Pawlowski, M. Reichert, Local quantum gravity. Phys. Rev. D92, 121501 (2015)

73. T. Denz, J.M. Pawlowski, M. Reichert, Towards apparent convergence in asymptotically safe quantum gravity. Eur. Phys. J. C78(4), 336 (2018)

74. A. Baldazzi, R.B.A. Zinati, K. Falls, Essential renormalisation group. SciPost Phys. 13, 085 (2022)

75. M. Becker, M. Reuter, Background independent field quantization with sequences of gravity-coupled approximants. Phys. Rev. D 102(12), 125001 (2020)

76. M. Becker, M. Reuter, Background independent field quantization with sequences of gravity-coupled approximants. II. Metric fluctuations. Phys. Rev. D 104(12), 125008 (2021)

77. G. 't Hooft, M.J.G. Veltman, Diagrammar. NATO Sci. Ser. B 4, 177-322 (1974)

78. A. Baldazzi, K. Falls, Essential quantum Einstein gravity. Universe 7(8), 294 (2021)

79. B. Knorr, Safe Essential Scalar-Tensor Theories, vol. 4 (2022). arXiv:2204.08564

80. R. Banerjee, M. Becker, R. Ferrero, N-cutoff regularization for fields on hyperbolic space, vol. 2 (2023). arXiv:2302.03547

81. W. Souma, Nontrivial ultraviolet fixed point in quantum gravity. Prog. Theor. Phys. 102, 181- 195 (1999)

82. M. Reuter, F. Saueressig, Renormalization group flow of quantum gravity in the Einstein-Hilbert truncation. Phys. Rev. D65, 065016 (2002)

83. O. Lauscher, M. Reuter, Ultraviolet fixed point and generalized flow equation of quantum gravity. Phys. Rev. D65, 025013 (2002)

84. D.F. Litim, Fixed points of quantum gravity. Phys. Rev. Lett. 92, 201301 (2004)

85. H. Gies, B. Knorr, S. Lippoldt, Generalized parametrization dependence in quantum gravity. Phys. Rev. D92(8), 084020 (2015)

86. D. Benedetti, K. Groh, P.F. Machado, F. Saueressig, The Universal RG machine. JHEP 1106, 079 (2011)

87. K. Groh, S. Rechenberger, F. Saueressig, O. Zanusso, Higher derivative gravity from the Universal renormalization group machine. PoS EPS-HEP2011, 124 (2011)

88. E.V. Gorbar, I.L. Shapiro, Renormalization group and decoupling in curved space. JHEP 02, 021 (2003)

89. E.V. Gorbar, I.L. Shapiro, Renormalization group and decoupling in curved space. II. The standard model and beyond. JHEP 06, 004 (2003)

90. Y. Decanini, A. Folacci, Off-diagonal coefficients of the Dewitt-Schwinger and Hadamard representations of the Feynman propagator. Phys. Rev. D73, 044027 (2006)

91. A. Codello, O. Zanusso, On the non-local heat Kernel expansion. J. Math. Phys. 54, 013513 (2013)

92. D. Brizuela, J.M. Martin-Garcia, G.A. Mena Marugan, xPert: computer algebra for metric perturbation theory. Gen. Rel. Grav. 41, 2415-2431 (2009)

93. J.W. York Jr., Conformatlly invariant orthogonal decomposition of symmetric tensors on Riemannian manifolds and the initial value problem of general relativity. J. Math. Phys. 14, 456-464 (1973)

94. D.V. Vassilevich, Heat kernel expansion: user's manual. Phys. Rept. 388, 279-360 (2003)

95. M. Reuter, H. Weyer, Quantum gravity at astrophysical distances? JCAP 0412, 001 (2004)

96. G. Gubitosi, R. Ooijer, C. Ripken, F. Saueressig, Consistent early and late time cosmology from the RG flow of gravity. JCAP 1812(12), 004 (2018)

97. A. Codello, R. Percacci, C. Rahmede, Ultraviolet properties of f(R)-gravity. Int. J. Mod. Phys. A23, 143-150 (2008)

98. P.F. Machado, F. Saueressig, On the renormalization group flow of f(R)-gravity. Phys. Rev. D77, 124045 (2008)

99. K. Falls, D. Litim, K. Nikolakopoulos, and C. Rahmede, A Bootstrap Towards Asymptotic Safety (2013). arXiv:1301.4191

100. K. Falls, C.R. King, D.F. Litim, K. Nikolakopoulos, C. Rahmede, Asymptotic safety of quantum gravity beyond Ricci scalars. Phys. Rev. D97(8), 086006 (2018)

101. K.G. Falls, D.F. Litim, J. Schröder, Aspects of asymptotic safety for quantum gravity. Phys. Rev. D 99(12), 126015 (2019)

102. H. Gies, B. Knorr, S. Lippoldt, F. Saueressig, Gravitational two-loop counterterm is asymptotically safe. Phys. Rev. Lett. 116(21), 211302 (2016)

103. U. Harst, M. Reuter, The 'Tetrad only' theory space: nonperturbative renormalization flow and asymptotic safety. JHEP 05, 005 (2012)

104. P. Dona, R. Percacci, Functional renormalization with fermions and tetrads. Phys. Rev. D87(4), 045002 (2013)

105. U. Harst, M. Reuter, A new functional flow equation for Einstein-Cartan quantum gravity. Ann. Phys. 354, 637-704 (2015)

106. U. Harst, M. Reuter, On selfdual spin-connections and asymptotic safety. Phys. Lett. B 753, 395-400 (2016)

107. C. Pagani, R. Percacci, Quantum gravity with torsion and non-metricity. Class. Quant. Grav. 32(19), 195019 (2015)

108. H. Gies, A.S. Salek, Asymptotically safe Hilbert-Palatini gravity in an on-shell reduction scheme. Eur. Phys. J. C 83(2), 173 (2023)

109. E. Manrique, S. Rechenberger, F. Saueressig, Asymptotically safe Lorentzian gravity. Phys. Rev. Lett. 106, 251302 (2011)

110. J. Biemans, A. Platania, F. Saueressig, Quantum gravity on foliated spacetimes: asymptotically safe and sound. Phys. Rev. D95(8), 086013 (2017)

111. J. Biemans, A. Platania, F. Saueressig, Renormalization group fixed points of foliated gravity-matter systems. JHEP 05, 093 (2017)

112. W.B. Houthoff, A. Kurov, F. Saueressig, Impact of topology in foliated Quantum Einstein Gravity. Eur. Phys. J. C77, 491 (2017)

113. A. Eichhorn, On unimodular quantum gravity. Class. Quant. Grav. 30, 115016 (2013)

114. A. Eichhorn, The renormalization group flow of unimodular f(R) gravity. JHEP 1504, 096 (2015)

115. R. Percacci, Unimodular quantum gravity and the cosmological constant. Found. Phys. 48(10), 1364-1379 (2018)

116. G.P. de Brito, A.D. Pereira, A.F. Vieira, Exploring new corners of asymptotically safe unimodular quantum gravity. Phys. Rev. D 103(10), 104023 (2021)

117. G.P. de Brito, O. Melichev, R. Percacci, A.D. Pereira, Can quantum fluctuations differentiate between standard and unimodular gravity? JHEP 12, 090 (2021)

118. J. Ambjorn, A. Görlich, J. Jurkiewicz, A. Kreienbuehl, R. Loll, Renormalization group flow in CDT. Class. Quant. Grav. 31, 165003 (2014)

119. A. Eichhorn, M. Schiffer, Asymptotic Safety of Gravity with Matter, vol. 12 (2022). arXiv:2212.07456